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Boundedness and Stability for a nonlinear difference equation with multiple delay

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Abstract. The equi-boundedness of solutions and the stability of the zero of nonlinear difference equation with bounded multiple delay

$$x_{n+1} = \lambda_n x_n + \sum_{i=1}^{r} \alpha_n^i F(x_{n-m_n}), \quad n = 0, 1, \cdots$$

are investigated.
Keywork: stability, fixed point theorem, contraction mapping, nonlinear difference equation, equi-boundedness.

1. Introduction

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{Z}$ the set of integers and $\mathbb{Z}^+$ the set of positive integers numbers. In this paper, we study the equi-boundedness of solutions and the stability of the zero of nonlinear difference equation with bounded multiple delay

$$x_{n+1} = \lambda_n x_n + \sum_{i=1}^{r} \alpha_n^i F(x_{n-m_n}), \quad n = 0, 1, \cdots \tag{1.1}$$

where α^i for $i = 1, 2, \cdots, r$ and λ are functions mapping $\mathbb{Z}$ to $\mathbb{R}$; F maps $\mathbb{R}$ to $\mathbb{R}$; m maps $\mathbb{Z}$ to $\mathbb{Z}^+$.

The properties of solutions of delay nonlinear difference equations has been studied extensively in recent years; see for example the work in [1-6] and the references cited therein. In [1], [2] and [3], the authors studied the oscillation and the asymptotic behaviour of solutions of the following nonlinear difference equations

$$x_{n+1} - x_n + \alpha(n) x_{n-m} = 0, \quad n = 0, 1, 2, \cdots$$

$$x_{n+1} - x_n + \sum_{i=1}^{r} \alpha_i(n) x_{n-m_i} = 0, \quad n = 0, 1, 2, \cdots$$

$$x_{n+1} - x_n + \alpha(n) f(x_{n-m}) = 0, \quad n = 0, 1, 2, \cdots$$

and

$$x_{n+1} = \lambda_n x_n + \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}).$$

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It is clear that these equations are particular cases of (1.1). We are particularly motivated by the work of the authors [1-6] on the stability, boundedness and convergence of solutions of difference equations.

Throughout this paper, we assume that there is a $K > 0$ so that if $|x| \leqslant K$ then $F(x) \leqslant K|x|$.

If m is bounded and the maximum of m is k, then for any integer $n_0 \geqslant 0$, we define $\mathbb{Z}_0$ to be the set of integers in $[n_0 - k, n_0]$. If m is unbounded then $\mathbb{Z}_0$ will be the set of integers in $(-\infty, n_0]$.

Let $\psi : \mathbb{Z}_0 \longrightarrow \mathbb{R}$ be an inital discrete bounded function.

We say $x_n := x_{n,n_0,\psi}$ is a solution of (1.1) if $x_n = \psi_n$ on $\mathbb{Z}_0$ and satisfies (1.1) for $n \geq n_0$.

The zero solution of (1.1) is Liapunov stable if for any $\epsilon > 0$ and any integer $n_0 \geq 0$ there exists a $\delta > 0$ such that $|\psi_n| \leqslant \delta$ on $\mathbb{Z}_0$ implies $|x_{n,n_0,\psi}| \leqslant \epsilon$ for $n \geq n_0$.

The zero solution of (1.1) is asymptotically stable if it is Liapunov stable and if for any integer $n_0 \geq 0$ there exists $r(n_0) > 0$ such that $|\psi_n| \leqslant r(n_0)$ on $\mathbb{Z}_0$ implies $|x_{n,n_0,\psi}| \to 0$ as $n \to \infty$.

A solution $x_n := x_{n,n_0,\psi}$ of (1.1) is said to be bounded if there exists a $B(n_0, \psi) > 0$ such that $|x_{n,n_0,\psi}| \leqslant B(n_0, \psi)$ for $n \geq n_0$.

A solution of (1.1) is said to be equi-bounded if for any n_0 and any $B_1 > 0$ there exists $B_2 = B_2(n_0, B_1) > 0$ such that $|\psi_n| \leqslant B_1$ on $\mathbb{Z}_0$ implies $|x_{n,n_0,\psi}| \leqslant B_2$ for $n \geq n_0$.

For any sequence $\{x_k\}$, we denote: $\sum\limits_{k=a}^{b} x_k = 0$, $\prod\limits_{k=a}^{b} x_k = 1$ for any $a > b$.

2. Main results

2.1. The Boundedness

Lemma 1. *Assume that $\lambda_n \neq 0$ for all $n \in \mathbb{Z}$. Then $\{x_n\}$ is a solution of equation (1.1) if and only if*

$$x_n = x_{n_0} \prod_{s=n_0}^{n-1} \lambda_s + \sum_{t=n_0}^{n-1} \sum_{i=1}^{r} \alpha_t^i F(x_{t-m_t}) \prod_{s=t+1}^{n-1} \lambda_s.$$

Proof. We first prove that equation (1.1) is equivalent to the equation

$$\Delta\Big(x_n \prod_{s=n_0}^{n-1} \lambda_s^{-1}\Big) = \sum_{i=1}^{r} \alpha_n^i F(x_{n-m_n}) \prod_{s=n_0}^{n} \lambda_s^{-1}. \tag{1.2}$$

Indeed, we have

$$\begin{aligned} x_{n+1} \prod_{s=n_0}^{n} \lambda_s^{-1} &= \lambda_n x_n \prod_{s=n_0}^{n} \lambda_s^{-1} + \sum_{i=1}^{r} \alpha_n^i F(x_{n-m_n}) \prod_{s=n_0}^{n} \lambda_s^{-1} \\ x_{n+1} \prod_{s=n_0}^{n} \lambda_s^{-1} &= x_n \prod_{s=n_0}^{n-1} \lambda_s^{-1} + \sum_{i=1}^{r} \alpha_n^i F(x_{n-m_n}) \prod_{s=n_0}^{n} \lambda_s^{-1}, \end{aligned}$$

or

$$\Delta\Big(x_n \prod_{s=n_0}^{n-1} \lambda_s^{-1}\Big) = \sum_{i=1}^{r} \alpha_n^i F(x_{n-m_n}) \prod_{s=n_0}^{n} \lambda_s^{-1}.$$

Now, summing equation (1.2) from n_0 to $n-1$ gives

$$\sum_{t=n_0}^{n-1} \Delta\Big(x_t \prod_{s=n_0}^{t-1} \lambda_s^{-1}\Big) = \sum_{t=n_0}^{n-1} \sum_{i=1}^{r} \alpha_t^i F(x_{t-m_t}) \prod_{s=n_0}^{t} \lambda_s^{-1}$$

$$x_n \prod_{s=n_0}^{n-1} \lambda_s^{-1} = x_{n_0} + \sum_{t=n_0}^{n-1} \sum_{i=1}^{r} \alpha_t^i F(x_{t-m_t}) \prod_{s=n_0}^{t} \lambda_s^{-1}$$

$$x_n = x_{n_0} \prod_{s=n_0}^{n-1} \lambda_s + \sum_{t=n_0}^{n-1} \sum_{i=1}^{r} \alpha_t^i F(x_{t-m_t}) \prod_{s=t+1}^{n-1} \lambda_s.$$

Theorem 1. *Assume that* $\lambda_n \neq 0$ *for* $n \geq n_0$ *and there exists* $M \in (0, +\infty)$, $\alpha \in (0,1)$ *such that*

$$\left| \prod_{s=n_0}^{n-1} \lambda_s \right| \leqslant M$$

and

$$\sum_{t=n_0}^{n-1} \left| \sum_{i=1}^{r} \alpha_t^i \right| \left| \prod_{s=t+1}^{n-1} \lambda_s \right| \leqslant \alpha, n \geqslant n_0. \tag{1.3}$$

Then solutions of (1.1) are equi-bounded.

Proof. Let B_1 be a positive constant. Choose $B_2 > 0$ such that

$$MB_1 + \alpha B_2^2 \leqslant B_2. \tag{1.4}$$

Let ψ be a bounded initial function satisfies $|\psi_n| \leqslant B_1$ on $\mathbb{Z}_0$. Define

$$S = \{\varphi : \mathbb{Z} \longrightarrow \mathbb{R} | \varphi_n = \psi_n \text{ on } \mathbb{Z}_0 \text{ and } ||\varphi|| \leqslant B_2\},$$

where $\|\varphi\| = \max_{n\in} |\varphi_n|$. We shall prove that $(S, ||.||)$ is a complete metric space.

+ $||.||$ is a metric.

i) $\forall \varphi, \eta \in S : \|\varphi - \eta\| = \max_{n\in} |(\varphi-\eta)_n| \geqslant 0,$

$$\begin{aligned} \|\varphi - \eta\| = 0 \quad &\Leftrightarrow \quad \max_{n\in} |(\varphi-\eta)_n| = 0 \\ &\Leftrightarrow \quad (\varphi-\eta)_n = 0, \forall n \in \mathbb{Z} \\ &\Leftrightarrow \quad \varphi_n - \eta_n = 0, \forall n \in \mathbb{Z} \\ &\Leftrightarrow \quad \varphi_n = \eta_n, \forall n \in \mathbb{Z} \\ &\Leftrightarrow \quad \varphi \equiv \eta. \end{aligned}$$

ii) $\forall \varphi, \eta \in S$, we have

$$\begin{aligned} \|\varphi - \eta\| &= \max_{n\in} |(\varphi-\eta)_n| = \max_{n\in} |\varphi_n - \eta_n| \\ &= \max_{n\in} |\eta_n - \varphi_n| = \max_{n\in} |(\eta-\varphi)_n| = \|\eta - \varphi\| . \end{aligned}$$

iii) $\forall \varphi, \eta, \psi \in S$, we have

$$\begin{aligned} \|\varphi-\eta\| &= \max_{n\in}|(\varphi-\psi)_n| = \max_{n\in}|\varphi_n-\psi_n| = \max_{n\in}|\varphi_n-\eta_n+\eta_n-\psi_n| \\ &\leqslant \max_{n\in}(|\varphi_n-\eta_n|+|\eta_n-\psi_n|) \leqslant \max_{n\in}|\varphi_n-\eta_n| + \max_{n\in}|\eta_n-\psi_n| \\ &= \|\varphi-\eta\|+\|\eta-\psi\|. \end{aligned}$$

+ Suppose that $\{\varphi^\ell\}$ is a Cauchy sequence in S. We have
$\forall\varepsilon>0, \exists\ell_0 : \forall k, \ell \geqslant \ell_0 : \left\|\varphi^\ell-\varphi^k\right\| < \varepsilon$
or $\forall\varepsilon>0, \exists\ell_0 : \forall k, \ell \geqslant \ell_0 : \max_{n\in}\left|\left(\varphi^\ell-\varphi^k\right)_n\right| < \varepsilon$
or

$$\forall\varepsilon>0, \exists\ell_0 : \forall k, \ell \geqslant \ell_0 : \left|\left(\varphi^\ell-\varphi^k\right)_n\right| < \varepsilon, \forall n \in \mathbb{Z}.$$

Fixed n, $\{\varphi_n^\ell\}$ is a Cauchy sequence in $\mathbb{R}$. In view of $\mathbb{R}$ is a complete metric space,

$$\exists\varphi_n \in \mathbb{R} : \varphi_n = \lim_{\ell\to\infty}\varphi_n^\ell.$$

We prove $\varphi \in S$. Indeed, since $\varphi^\ell \in S$, $\varphi_n^\ell = \psi_n$ on $\mathbb{Z}_0$. It implies $\varphi_n = \lim_{\ell\to\infty}\varphi_n^\ell = \psi_n$ on $\mathbb{Z}_0$. Moreover, since $||\varphi^\ell|| \leqslant B_2$, $||\varphi|| \leqslant B_2$.

Define mapping $P : S \longrightarrow S$ by $(P\varphi)_n = \psi_n$ on $\mathbb{Z}_0$ and

$$(P\varphi)_n = \psi_{n_0}\prod_{s=n_0}^{n-1}\lambda_s + \sum_{t=n_0}^{n-1}\sum_{i=1}^{r}\alpha_t^i F(\varphi_{t-m_t})\prod_{s=t+1}^{n-1}\lambda_s, n \geqslant n_0. \tag{1.5}$$

We first prove that P maps from S to S. Indeed, we have

$$\begin{aligned} |(P\varphi)_n| &= \left|\psi_{n_0}\prod_{s=n_0}^{n-1}\lambda_s + \sum_{t=n_0}^{n-1}\sum_{i=1}^{r}\alpha_t^i F(\varphi_{t-m_t})\prod_{s=t+1}^{n-1}\lambda_s\right|, n \geqslant n_0 \\ &\leqslant |\psi_{n_0}|\left|\prod_{s=n_0}^{n-1}\lambda_s\right| + \sum_{t=n_0}^{n-1}\left|\sum_{i=1}^{r}\alpha_t^i\right| |F(\varphi_{t-m_t})| \left|\prod_{s=t+1}^{n-1}\lambda_s\right|, n \geqslant n_0. \end{aligned}$$

Since $||\varphi|| \leqslant B_2, |\varphi_{t-m_t}| \leqslant B_2$. So $F(\varphi_{t-m_t}) \leqslant B_2||\varphi_{t-m_t}|| \leqslant B_2^2$. Hence,

$$\begin{aligned} |(P\varphi)_n| &\leqslant B_1M + B_2^2\sum_{t=n_0}^{n-1}\left|\sum_{i=1}^{r}\alpha_t^i\right|\left|\prod_{s=t+1}^{n-1}\lambda_s\right|, n \geqslant n_0 \\ &\leqslant B_1M + B_2^2\sum_{t=n_0}^{n-1}\left|\sum_{i=1}^{r}\alpha_t^i\right|\left|\prod_{s=t+1}^{n-1}\lambda_s\right|, n \geqslant n_0 \\ &\leqslant B_1M + \alpha B_2^2 \leqslant B_2. \end{aligned}$$

Hence P maps from S to itself. We next show that P is a contraction under the supremum norm. Let $\varphi, \eta \in S$, we get

$$\begin{aligned} |(P\varphi)_n - (P\eta)_n| &= \left|\sum_{t=n_0}^{n-1}\sum_{i=1}^{r}\alpha_t^i F(\varphi_{t-m_t})\prod_{s=t+1}^{n-1}\lambda_s - \sum_{t=n_0}^{n-1}\sum_{i=1}^{r}\alpha_t^i F(\eta_{t-m_t})\prod_{s=t+1}^{n-1}\lambda_s\right| \\ &\leqslant B_2\sum_{t=n_0}^{n-1}\left|\sum_{i=1}^{r}\alpha_t^i\right|\left|\prod_{s=t+1}^{n-1}\lambda_s\right|\|\varphi-\eta\| \\ &\leqslant B_2\alpha\|\varphi-\eta\|. \end{aligned}$$

Next, we prove that $B_2\alpha \in (0,1)$. Indeed, since $\frac{MB_1}{B_2} > 0$, $1 - \frac{MB_1}{B_2} < 1$. On the other hand, from $MB_1 + \alpha B_2^2 \leqslant B_2$ we have $\alpha B_2^2 \leqslant B_2 - MB_1$, which implies that

$$\alpha B_2 \leqslant 1 - \frac{MB_1}{B_2} < 1.$$

This shows that P is a contraction. Thus, by the contraction mapping principle, P has a unique fixed point $\varphi^* \in S$. We have

$$(P\varphi^*)_n = \varphi_n^* = \psi_{n_0}\prod_{s=n_0}^{n-1}\lambda_s + \sum_{t=n_0}^{n-1}\sum_{i=1}^{r}\alpha_t^i F(\varphi_{t-m_t}^*)\prod_{s=t+1}^{n-1}\lambda_s.$$

Since $n_0 \in \mathbb{Z}_0$ and $\varphi^* \in S$, $\psi_{n_0} = \varphi_{n_0}^*$. Hence

$$\varphi_n^* = \varphi_{n_0}^*\prod_{s=n_0}^{n-1}\lambda_s + \sum_{t=n_0}^{n-1}\sum_{i=1}^{r}\alpha_t^i F(\varphi_{t-m_t}^*)\prod_{s=t+1}^{n-1}\lambda_s,$$

i.e φ_n^* is a solution of (1.1). This prove that solutions of (1.1) are equi-bounded.

2.2. The Stability

Theorem 2. *Assume that there exists $\gamma \in (0,1)$ such that $|\sum_{i=1}^{r}\alpha_n^i| \leqslant \gamma$ and $|\lambda_n| < 1-\gamma$ for all $n \in \mathbb{Z}$. Then the zero solution of (1.1) is Liapunov stable.*

Proof. Put

$$M = (1-\gamma)(n-n_0), N = (1-\gamma)^{n-t-1}, \alpha = \gamma(n-n_0)N.$$

We have

$$\begin{aligned}
\sum_{s=n_0}^{n-1} |\lambda_s| &< \sum_{s=n_0}^{n-1} (1-\gamma) = (1-\gamma)(n-n_0) = M \\
\prod_{s=t+1}^{n-1} \lambda_s &< \prod_{s=t+1}^{n-1} (1-\gamma) = (1-\gamma)^{n-t-1} = N \\
|\sum_{s=n_0}^{n-1} \lambda_s| &\leqslant \sum_{s=n_0}^{n-1} |\lambda_s| < M \\
\sum_{s=n_0}^{n-1} |\sum_{i=1}^{r} \alpha_s^i| &\leqslant \gamma(n-n_0) \\
\sum_{s=n_0}^{n-1} |\sum_{i=1}^{r} \alpha_s^i|| \prod_{s=t+1}^{n-1} \lambda_s| &\leqslant \gamma(n-n_0)N = \alpha.
\end{aligned}$$

Let $\epsilon > 0$. Choose $\delta > 0$ such that

$$M\delta + \alpha\epsilon^2 \leqslant \epsilon.$$

Let ψ be a bounded initial function satisfies $|\psi_n| \leqslant \delta$ on $\mathbb{Z}_0$. Define

$$S = \{\varphi : \mathbb{Z} \longrightarrow \mathbb{R} | \varphi_n = \psi_n \text{ on } \mathbb{Z}_0 \text{ and } ||\varphi|| \leqslant \epsilon\},$$

where $\|\varphi\| = \max_{n\in Z} |\varphi_n|$. It can be verified that $(S, ||.||)$ is a complete metric space.

Consider the map $P : S \longrightarrow S$ by (1.5). We have

$$\begin{aligned}
|(P\varphi)_n| &= \left| \psi_{n_0} \prod_{s=n_0}^{n-1} \lambda_s + \sum_{t=n_0}^{n-1} \sum_{i=1}^{r} \alpha_t^i F(\varphi_{t-m_t}) \prod_{s=t+1}^{n-1} \lambda_s \right|, n \geqslant n_0 \\
&\leqslant |\psi_{n_0}| \left| \prod_{s=n_0}^{n-1} \lambda_s \right| + \sum_{t=n_0}^{n-1} \left| \sum_{i=1}^{r} \alpha_t^i \right| |F(\varphi_{t-m_t})| \left| \prod_{s=t+1}^{n-1} \lambda_s \right|, n \geqslant n_0 \\
&\leqslant \delta M + \varepsilon^2 \sum_{t=n_0}^{n-1} \left| \sum_{i=1}^{r} \alpha_t^i \right| \left| \prod_{s=t+1}^{n-1} \lambda_s \right|, n \geqslant n_0 \\
&\leqslant \delta M + \varepsilon^2 \alpha < \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
|(P\varphi)_n - (P\eta)_n| &= \left| \sum_{t=n_0}^{n-1} \sum_{i=1}^{r} \alpha_t^i F(\varphi_{t-m_t}) \prod_{s=t+1}^{n-1} \lambda_s - \sum_{t=n_0}^{n-1} \sum_{i=1}^{r} \alpha_t^i F(\eta_{t-m_t}) \prod_{s=t+1}^{n-1} \lambda_s \right| \\
&\leqslant \varepsilon \left| \sum_{t=n_0}^{n-1} \sum_{i=1}^{r} \alpha_t^i \prod_{s=t+1}^{n-1} \lambda_s \right| \|\varphi - \eta\| \leqslant \varepsilon \sum_{t=n_0}^{n-1} \left| \sum_{i=1}^{r} \alpha_t^i \right| \left| \prod_{s=t+1}^{n-1} \lambda_s \right| \|\varphi - \eta\| \\
&\leqslant \varepsilon\alpha \|\varphi - \eta\|.
\end{aligned}$$

It is easy to check that $\alpha\varepsilon \in (0, 1)$. This show that P is a contraction map and for any $\varphi \in S, ||P\varphi|| \leqslant \epsilon$. Therefore the zero solution of (1.1) Liapunov stable.

Theorem 3. *Assume that the hypotheses of Theorem 2 are satisfied. Assume, in addition, that*

$$n - m_n \to \infty \text{ as } n \to \infty. \tag{1.6}$$

Then the zero solution of (1.1) is asymptotically stable.

Proof. Since $|\lambda_n| < 1 - \gamma$ for all $n \in \mathbb{Z}$ and $\gamma \in (0, 1)$, it follows that $|\lambda_n| < 1$. Consequently,

$$\prod_{s=n_0}^{n-1} \lambda_s \to 0 \text{ as } n \to \infty. \tag{1.7}$$

Let ψ be a bounded initial function satisfies $|\psi_n| \leqslant r(n_0)$. Define

$$S^* = \{\varphi : \mathbb{Z} \longrightarrow \mathbb{R} | \varphi_n = \psi_n \text{ on } \mathbb{Z}_0, ||\varphi|| \leqslant \varepsilon \text{ and } |\varphi_n| \to 0, \text{ as } n \to \infty\}.$$

Define $P : S^* \longrightarrow S^*$ by (1.5). From the proof of Theorem 2, the map P is a contraction and it maps from S^* to itself.

We next prove that $(P\varphi)_n$ goes to zero as n goes to infinity.

Since (1.7), it follows that $\psi_{n_0} \prod\limits_{s=n_0}^{n-1} \lambda_s$ goes to zero as n goes to infinie. We have only to prove $\sum\limits_{t=n_0}^{n-1} \sum_{i=1}^{r} \alpha_t^i F(\varphi_{t-m_t}) \prod\limits_{s=t+1}^{n-1} \lambda_s, (n \geqslant n_0) \longrightarrow 0$ as $n \longrightarrow \infty$. Let $\varphi \in S^*$ then $|\varphi_{n-m_n}| \leqslant \varepsilon$. Also, since $\varphi_{n-m_n} \to 0$ as $n - m_n \to \infty$, there exists a $n_1 > 0$ such that for $n > n_1$, $|\varphi_{n-m_n}| < \varepsilon_1$ for $\varepsilon_1 > 0$.

Indeed, by the condition (1.7), there exists $n_2 > n_1$ such that

$$\left| \prod_{s=n_1}^{n} \lambda_s \right| < \frac{\varepsilon_1}{\alpha \varepsilon^2} \quad \forall n > n_2.$$

Hence, for all $n > n_2$, we have

$$\begin{aligned}
&\left| \sum_{t=n_0}^{n-1} \sum_{i=1}^{r} \alpha_t^i F(\varphi_{t-m_t}) \prod_{s=t+1}^{n-1} \lambda_s \right| \leqslant \sum_{t=n_0}^{n-1} \left| \sum_{i=1}^{r} \alpha_t^i F(\varphi_{t-m_t}) \prod_{s=t+1}^{n-1} \lambda_s \right| \\
&\leqslant \sum_{t=n_0}^{n_1-1} \left| \sum_{i=1}^{r} \alpha_t^i F(\varphi_{t-m_t}) \prod_{s=t+1}^{n-1} \lambda_s \right| + \sum_{t=n_1}^{n-1} \left| \sum_{i=1}^{r} \alpha_t^i F(\varphi_{t-m_t}) \prod_{s=t+1}^{n-1} \lambda_s \right| \\
&\leqslant \epsilon^2 \sum_{t=n_0}^{n_1-1} \left| \sum_{i=1}^{r} \alpha_t^i \prod_{s=t+1}^{n-1} \lambda_s \right| + \epsilon_1^2 \sum_{t=n_1}^{n-1} \left| \sum_{i=1}^{r} \alpha_t^i \prod_{s=t+1}^{n-1} \lambda_s \right| \\
&\leqslant \epsilon^2 \sum_{t=n_0}^{n_1-1} \left| \sum_{i=1}^{r} \alpha_t^i \prod_{s=t+1}^{n_1-1} \lambda_s \prod_{s=n_1}^{n-1} \lambda_s \right| + \epsilon_1^2 \alpha \\
&\leqslant \varepsilon^2 \left| \prod_{s=n_1}^{n-1} \lambda_s \right| \sum_{t=n_0}^{n_1-1} \left| \sum_{i=1}^{r} \alpha_t^i \right| \left| \prod_{s=t+1}^{n_1-1} \lambda_s \right| + \varepsilon_1 \alpha \\
&\leqslant \varepsilon^2 \alpha \left| \prod_{s=n_1}^{n-1} \lambda_s \right| + \varepsilon_1^2 \alpha \leqslant \varepsilon^2 \alpha \cdot \frac{\varepsilon_1}{\varepsilon^2 \alpha} + \varepsilon_1^2 \alpha. \\
&\leqslant \varepsilon_1 + \varepsilon_1^2 \alpha.
\end{aligned}$$

Now, by the above, it follows that $(P\varphi)_n \to 0$ as $n \to \infty$. By the contraction mapping principle, P has a unique fixed point that solves (1.1) and goes to zero as n goes to infinity. The proof is complete.

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