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# Non-commutative chern characters of the $C^*$-algebras of the sphers

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**Abstract.** We propose in this paper the construcion of non-commutative Chern characters of the $C^*$−algebras of spheres and quantum spheres. The final computation gives us clear relation with the ordinary $\mathbb{Z}/(2)$−graded Chern chracters of torsion or their normalizers.
*Keyworks:* Characters of the $C^*$−algebras.

## 1. Introduction

For compact Lie groups the Chern character $ch : K^*(G) \otimes \mathbb{Q} \longrightarrow H^*_{DR}(G; \mathbb{Q})$ were constructed. In [4]-[5] we computed the non-commutative Chern characters of compact Lie goup $C^*$−algebras and of compact quantum groups, which are also homomorphisms from quantum $K$−groups into entire current periodic cyclic homology of group $C*$−algebras (resp., of $C^*$−algebra quantum groups), $ch_{C^*} : K_*(C^*(G)) \longrightarrow HE_*(C^*(G)))$, (resp., $ch_{C^*} : K_*(C^*_\varepsilon(G)) \longrightarrow HE_*(C^*_\varepsilon(G)))$. We obtained also the corresponding algebraic vesion $ch_{alg} : K_*(C^*(G)) \longrightarrow HP_*(C^*(G)))$, which coincides with the Fedosov-Cuntz- Quillen formula for Chern characters [5]. When $A = C^*_\varepsilon(G)$ we first computed the $K$−groups of $C^*_\varepsilon(G)$ and the $HE_*(C^*_\varepsilon(G))$. Thereafter we computed the Chern character $ch_{C^*} : K_*(C^*_\varepsilon(G)) \longrightarrow HE_*(C^*_\varepsilon(G)))$ as an isomorphism modulo torsions.

Using the results from [4]-[5], in this paper we compute the non-commutative Chern characters $ch_{C^*} : K_*(A) \longrightarrow HE_*(A)$, for two cases $A = C^*(S^n))$, the $C^*$−algebra of spheres and $A = C^*_\varepsilon(S^n)$, the $C^*$−algebras of quantum spheres. For compact groups $G = O(n+1)$, the Chern character $ch : K^*(S^n) \otimes \mathbb{Q} \longrightarrow H^*_{DR}(S^n; \mathbb{Q})$ of the sphere $S^n = O(n+1)/O(n)$ is an isomorphism (se, [15]). In the paper, we describe two Chern character homomorphisms

$$ch_{C^*} : K_*(C^*(S^n)) \longrightarrow HE_*(C^*(S^n)),$$

and

$$ch_{C^*} : K_*(C^*_\varepsilon(S^n)) \longrightarrow HE_*(C^*_\varepsilon(S^n)).$$

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Finally, we show that there is a commutative diagram

$$\begin{array}{ccc} K_*(C^*(S^n)) & \xrightarrow{ch_{C^*}} & HE_*(C^*(S^n)) \\ \cong\downarrow & & \downarrow\cong \\ K_*(\mathbb{C}(\mathcal{N}_{T_n})) & \xrightarrow{ch_{C^*}} & HE_*(\mathbb{C}(\mathcal{N}_{T_n})) \\ \cong\downarrow & & \downarrow\cong \\ K^*(\mathcal{N}_{T_n}) & \xrightarrow{ch} & HE^*_{DR}(\mathcal{N}_{T_n})) \end{array}$$

(Similarly, for $A = C^*_\varepsilon(^n)$, we have an analogous commutative diagram with $W \times S^1$ of place of $W \times S^n$), from which we deduce that $ch_{C^*}$ is an isomorphism modulo torsions.

We now briefly review he structure of the paper. In section 1, we compute the Chern chracter of the $C^*-$algebras of spheres. The computation of Chern character of $C^*(S^n)$ is based on two crucial points:

i) Because the sphere $S^n = O(n+1)/O(n)$ is a homogeneous space and $C^*-$algebra of $S^n$ is the transformation group $C^*-$algebra, follwing J.Parker [10], we have:

$$C^*(S^n) \cong C^*((O(n)) \otimes \mathcal{K}(L^2(S^n))).$$

ii) Using the stability property theorem $K_*$ and $HE_*$ in [5], we reduce it to the computation of $C^*-$algebras of subgroup $O(n)$ in $O(n+1)$ group.

In section 2, we compute the Chern character of $C^*-$algebras of quantum spheres. For quantum sphere $S^n$, we define the compact quantum $C^*-$algebras $C^*_\varepsilon(S^n)$, where $\varepsilon$ is a positive real number. Thereafter, we prove that:

$$C^*_\varepsilon(S^n) \cong \mathbb{C}(S^1) \oplus \bigoplus_{e\neq\omega\in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{\omega,t})dt,$$

where $\mathcal{K}(H_{\omega,t})$ is the elementary algebra of compact operators in a separable infinite dimensional Hilbert space $H_{\omega,t}$ and $W$ is the Weyl of a maximal torus $\mathbf{T_n}$ in $SO(n)$.

Similar to Section 1, we first compute the $K_*(C^*_\varepsilon(S^n)$ and $HE_*(C^*_\varepsilon(S^n)$, and we prove that

$$ch_{C^*} : K_*(C^*_\varepsilon(S^n)) \longrightarrow HE_*(C^*_\varepsilon(S^n))$$

is a isomorphism modulo torsion.

**Notes on Notation:** For any compact space $X$, we write $K^*(X)$ for the $\mathbb{Z}/(2)-$ graded topological $K-$theory of $X$. We use Swan's theorem to identify $K^*(X)$ with $\mathbb{Z}/(2)-$ graded $K^*(\mathbb{C}(X))$. For any involution Banach algebra $A$, $K_*(A)$, $HE_*(A)$ and $HP_*(A)$ are $\mathbb{Z}/(2)-$ graded algebraic or topological $K-$groups of $A$, enire cyclic homology, and periodic cyclic homology of $A$, respectively. If $\mathbf{T}$ is a maximal torus of a compact group $G$, with the corresponding Weyl group $W$, write $\mathbf{C}(\mathbf{T})$ for the algebra of complex valued functions on $\mathbf{T}$. We use the standard notation from the root theory such as $P, P^+$ for the positive highest weights, etc... We denote by $\mathcal{N}_\mathbf{T}$ the normalizer of $\mathbf{T}$ in $G$, by $\mathbb{N}$ the set of natural numbers, $\mathbb{R}$ the fied of real numbers and $\mathbb{C}$ the field of complex numbers, $\ell^2_A(\mathbb{N})$ the standard $\ell^2$ space of square integrable sequences of elements from $A$, and finally by $C^*_\varepsilon(G)$ we denote the compact quantum algebras, $C^*(G)$ the $C^*-$algebra of $G$.

**2. Non-commutative Chern characters of $C^*-$algebras of spheres.**

In this section, we compute non-commutative Chern characters of $C^*-$algebras of spheres. Let $A$ be an involution Banach algebra. We construct the non-commutative Chern characters $ch_{C^*}$ : $K_*(A) \longrightarrow HE_*(A)$, and show in [4] that for $C^*-$algebra $C^*(G)$ of compact Lie groups $G$, the Chern character $ch_{C^*}$ is an isomorphism.

**Proposition 2.1.** ([5], Theorem 2.6) *Let $H$ be a separable Hilbert space and $B$ an arbitrary Banach space. We have*

$$\begin{aligned} K_*(\mathcal{K}(H)) &\cong K_*(\mathbb{C}); \\ K_*(B \otimes \mathcal{K}(H)) &\cong K_*(B) \\ HE_*(\mathcal{K}(H)) &\cong HE_*(\mathbb{C}); \\ HE_*(B \otimes \mathcal{K}(H)) &\cong HE_*(B), \end{aligned}$$

*where $\mathcal{K}(H)$ is the elementary algebra of compact operators in a separable infinite dimensional Hilbert space $H$.*

**Proposition 2.2.** ([5], Theorem 3.1) *Let $A$ be an involution Banach algebra with unity. There is a Chern character homomorphism*

$$ch_{C^*} : K_*(A) \longrightarrow HE_*(A).$$

**Proposition 2.3.** ([5], Theorem 3.2) *Let $G$ be an compact group and $\mathbf{T}$ a fixed maximal torus of $G$ with Weyl $W := \mathcal{N}_{\mathbf{T}}/\mathbf{T}$. Then the Chern character $ch_{C^*} : K_*(C^*(G)) \longrightarrow HE_*(C^*(G))$. is an isomorphism modulo torsions. i.e.*

$$ch_{C^*} : K_*(C^*(G)) \otimes \mathbb{C} \xrightarrow{\cong} HE_*(C^*(G)),$$

*which can be identified with the classical Chern character*

$$ch_{C^*} : K_*(C(\mathcal{N}_{\mathbf{T}})) \longrightarrow HE_*(C(\mathcal{N}_{\mathbf{T}})),$$

*that is also an isomorphic modulo torsion, i.e*

$$ch : K_*(\mathcal{N}_{\mathbf{T}}) \otimes \mathbb{C} \xrightarrow{\cong} H^*_{DR}(\mathcal{N}_{\mathbf{T}}).$$

Now, for $S^n = O(n+1)/O(n)$, where $O(n), O(n+1)$ are the orthogonal matrix groups. We denote by $\mathbf{T}_n$ a fixed maximal torus of $O(n)$ and $\mathcal{N}_{\mathbf{T}_n}$ the normalizer of $\mathbf{T}_n$ in $O(n)$. Following Proposition 1.2, there a natural Chern character $ch_{C^*}$ : $K_*(C(S^n)) \longrightarrow HE_*(C(S^n))$. Now, we compute first $K_*(C(S^n))$ and then $HE_*(C(S^n))$ of $C^*-$algebra of the sphere $S^n$.

Proposition 2.4.

$$HE_*(C(S^n)) \cong H^W_{DR}(\mathbf{T}_n)).$$

*Proof.* We have

$$\begin{aligned} HE_*(C(S^n)) &= HE_*(C(O(n+1)/O(n))) \\ &\cong HE_*(C^*(O(n)) \otimes \mathcal{K}(L^2(O(n+1)/O(n)))) \end{aligned}$$

(in virtue of, the $\mathcal{K}(L^2(O(n+1)/O(n))))$ is a $C*-$algebra compact operators in a separable Hilbert space $L^2(O(n+1)/O(n)))$

$$\begin{aligned} &\cong HE_*(C(O(n))) \quad \text{(by Proposition 1.1)} \\ &\cong HE_*(\mathbb{C}(\mathcal{N}_{T_n})) \quad \text{(see [5])}. \end{aligned}$$

Thus, we have $HE_*(C^*(S^n)) \cong HE^*(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n}))$.

Apart from that, because $\mathbb{C}(\mathcal{N}_{\mathbf{T}_n})$ is then commutative $C^*-$algebra, by a Cuntz- Quillen's result [1], we have an isomorphism

$$HP_*(\mathbb{C}((N_{\mathbf{T}_n})) \cong H^*_{DR}(\mathcal{N}_{\mathbf{T}_n})).$$

Moreover, by a result of Khalkhali [8],[9], we have

$$HP_*(\mathbb{C}((N_{\mathbf{T}_n})) \cong HE_*(\mathbb{C}((N_{\mathbf{T}_n})).$$

We have, hence

$$\begin{aligned} HE_*(C^*(S^n)) &\cong HE^*(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n})) \cong HP_*(\mathbb{C}((N_{\mathbf{T}_n})) \\ &\cong H^*_{DR}(\mathcal{N}_{\mathbf{T}_n}) \cong H^W_{DR}(\mathcal{N}_{\mathbf{T}_n}) \quad \text{(by [15])}. \end{aligned}$$

**Remark 1.** Because $H^W_{DR}(\mathcal{N}_{\mathbf{T}_n})$ is the de Rham cohomology of $\mathbf{T}_n$, invariant under the action of the Weyl group $W$, following Watanabe [15], we have a canonical isomorphism $H^W_{DR}(\mathbf{T}_n) \cong H^*(SOn)) = \Lambda\ (x_3, x_7, ..., x_{2i+3})$, where $x_{2i+3} = \sigma^*(p_i) \in H^{2n+3}(S0(n))$ and $\sigma^* : H^*(BSO(n), R) \longrightarrow H^*(SO(n), R)$ for a commutative ring $R$ with a unit $1 \in R$, and $p_i = \sigma_i(t_1^2, t_2^2, ..., t_i^2) \in H*(B\mathbf{T}_n\mathbb{Z})$ the Pontryagin classes.

Thus, we have

$$HE_*(C^*(S^n)) = \Lambda\ (x_3, x_7, ..., x_{2i+3}).$$

**Proposition 2.5.**

$$K_*(C(S^n)) \cong K^*(\mathcal{H}_{\mathbf{T}_n})).$$

*Proof.* We have

$$\begin{aligned} K_*(C(S^n)) &= K_*(C(O(n+1)/O(n))) \\ &\cong K_*(C^*(O(n)) \otimes \mathcal{K}(L^2(O(n+1)/O(n)))) \quad \text{(see [10])} \\ &\cong K_*(C^*(O(n))) \quad \text{(by Proposition 1.1)} \\ &\cong K_*(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n})) \\ &\cong K_*(\mathcal{N}_{\mathbf{T}_n}) \quad \text{(by Lemma 3.3, from [5])}. \end{aligned}$$

Thus, $K_*(C(S^n)) \cong K_*(\mathcal{N}_{\mathbf{T}_n})$.

**Remark 2.** Following Lemma 4.2 from [5], we have

$$\begin{aligned} K_*(\mathcal{N}_{\mathbf{T}_n}) &\cong K^*(SO(n+1))/Tor \\ &= \Lambda\ (\beta(\lambda_1), ...., \beta(\lambda_{n-3}, \varepsilon_{n+1}), \end{aligned}$$

where $\beta : R(SO(n)) \longrightarrow \widetilde{K}^{-1}(SO(n))$ be the homomorphism of Abelian groups assigning to each representation $\rho : SO(n) \longrightarrow U(n+1)$ the homotopic class $\beta(\rho) = [i_n\rho] \in [SO(n), U] = \widetilde{K}^{-1}(SO(n))$, where $i_n : U(n+1) \longrightarrow U$ is the canonical one, $U(n+1)$ and $U$ by the $n-th$ and infinite unitary groups respectively and $\varepsilon_{n+1} \in K^{-1}(SO(n+1))$. We have, finally

$$K^*(C^*(S^n)) \cong \Lambda\ (\beta(\lambda_1), ...., \beta(\lambda_{n-3}, \varepsilon_{n+1}).$$

Moreover, the Chern character of $SU(n+1)$ was computed in [14], for all $n \geqslant 1$. Let us recall the result. Define a function

$$\phi : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Z},$$

given by

$$\phi(n,k,q) = \sum_{i=1}^{k} (-1)^{i-1} \binom{n}{k-1} i^{q-1}.$$

**Theorem 2.6.** *Let* $\mathbf{T}_n$ *be a fixed maximal torus of* $O(n)$ *and* $T$ *the fixed maximal torus of* $SO(n)$, *with Weyl groups* $W := \mathcal{N}_{\mathbf{T}}/\mathbf{T}$, *the Chern character of* $C^*(S^n)$

$$ch_{C^*} : K_*(C^*(S^n)) \longrightarrow HE_*(C^*(S^n))$$

*is an isomorphism, given by*

$$\begin{aligned} ch_{C^*}(\beta(\lambda_k)) &= \sum_{i=1}^{n} ((-1)^{i-1} 2/(2i-1)! \phi(2n+1,k,2i) x_{2i+1} \quad \text{(k=1, ..., n-1)}; \\ ch_{C^*}(\varepsilon_{n+1}) &= \sum_{i=1}^{n} ((-1)^{i-1} 2/(2i-1)!)((\frac{1}{2^n} \sum_{i=1}^{n} \phi(2n+1,k,2i) x_{2i+1}. \end{aligned}$$

*Proof.* By Proposition 1.5, we have

$$K_*(C^*(S^n)) \cong K_*(\mathbb{C}(\mathcal{N}_{T_n})) \cong K^*(\mathcal{N}_{T_n}$$

and

$$HE_*(C^*(S^n)) \cong HE_*(\mathbb{C}(\mathcal{N}_{T_n})) \cong H^*_{DR}(\mathcal{N}_{T_n}) \quad \text{(by Proposition 1.4).}$$

Now, consider the commutative diagram

$$\begin{array}{ccc} K_*(C^*(S^n)) & \xrightarrow{ch_{C^*}} & HE_*(C^*(S^n)) \\ \cong\downarrow & & \downarrow\cong \\ K_*(\mathbb{C}(\mathcal{N}_{T_n})) & \xrightarrow{ch_{CQ}} & HE_*(\mathbb{C}(\mathcal{N}_{T_n})) \\ \cong\downarrow & & \downarrow\cong \\ K^*(\mathcal{N}_{T_n}) & \xrightarrow{ch} & H^*_{DR}(\mathcal{N}_{T_n})). \end{array}$$

Moreover, by the results of Watanabe [15], the Chern character $ch : K^*(\mathcal{N}_{\mathbf{T}_n}) \otimes \mathbb{C} \longrightarrow H^*_{DR}(\mathcal{N}_{\mathbf{T}_n}))$ is an isomorphism

Thus, $ch_{C^*} : K_*(C^*(S^n)) \longrightarrow HE_*(C^*(S^n))$ is an isomorphic (Proposition 1.4 and 1.5), given by

$$\begin{aligned} ch_{C^*}(\beta(\lambda_k)) &= \sum_{i=1}^{n} ((-1)^{i-1} 2/(2i-1)!) \phi(2n+1,k,2i) x_{2i+1} \quad \text{(k=1, ..., n-1)}; \\ ch_{C^*}(\varepsilon_{n+1}) &= \sum_{i=1}^{n} ((-1)^{i-1} 2/(2i-1)!) \Big( (\frac{1}{2^n}) \sum_{i=1}^{n} \phi(2n+1,k,2i) \Big) x_{2i+1}, \end{aligned}$$

where

$$\begin{aligned} K^*(C^*(S^n)) &\cong \Lambda\ (\beta(\lambda_1), ...., \beta(\lambda_{n-3}, \varepsilon_{n+1}) \\ HE_*(C^*(S^n)) &\cong \Lambda\ (x_3, x_7, ..., x_{2i+3}). \end{aligned}$$

## 3. Non-commutative Chern characters of $C^*$−algebras of quantum spheres

In this section, we at first recall definition and main properties of compact quantum spheres and their representations. More precisely, for $S^n$, we define $C^*_\varepsilon(S^n)$, the $C^*$−algebras of compact quantum spheres as the $C^*$−completion of the $^*$−algebra $\mathcal{F}_\varepsilon(S^n)$ with respect to the $C^*$−norm, where $\mathcal{F}_\varepsilon(S^n)$ is the quantized Hopf subalgebra of the Hopf algebra, dual to the quantized universal enveloping algebra $U(\mathcal{G})$, generated by matrix elements of the $U(\mathcal{G})$ modules of type **1**(see [3]). We prove that

$$C^*_\varepsilon(S^n) \cong \mathbb{C}(S^1) \oplus \bigoplus_{e\neq\omega\in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{\omega,t})dt,$$

where $\mathcal{K}(H_{\omega,t})$ is the elementary algebra of compact operators in a separable infinite dimensional Hilbert space $H_{\omega,t}$ and $W$ is the Weyl group of $S^n$ with respect to a maximal torus $\mathbf{T}$.

After that,we first compute the $K$−groups $K_*(C^*_\varepsilon(S^n))$ and the $HE_*(C^*_\varepsilon(S^n))$, respectively. Thereafter we define the Chern character of $C^*$−algebras quantum spheres, as a homomorphism from $K_*(C^*_\varepsilon(S^n))$ to $HE_*(C^*_\varepsilon(S^n))$, and we prove that $ch_{C^*} : K_*(C^*_\varepsilon(S^n)) \longrightarrow HE_*(C^*_\varepsilon(S^n))$ is an isomorphism modulo torsion.

Let $G$ be a complex algebraic group with Lie algebra $\mathcal{G}$ =Lie$G$ and $\varepsilon$ is real number, $\varepsilon \neq -1$.

**Definition 3.1.** ([3], Definition 13.1). *The quantized function algebra $\mathcal{F}_\varepsilon(G)$ is the subalgebra of the Hopf algebra dual to $U_\varepsilon(\mathcal{G})$, generated by the matrix elements of the finite-dimensional $U_\varepsilon(\mathcal{G})$−modules of type* **1**.

For compact quantum groups the unitary representations of $\mathcal{F}_\varepsilon(G)$ are parameterized by pairs $(\omega, t)$, where $t$ is an element of a fixed maximal torus of the compact real form of $G$ and $\omega$ is a element of the Weyl group $W$ of $\mathbf{T}$ in $G$.

Let $\lambda \in P^+, V_\varepsilon(\lambda)$ be the irreducible $U_\varepsilon(\mathcal{G})$−module of type **1** with the highest weight $\lambda$. Then $V_\varepsilon(\lambda)$ admits a positive definite hermitian form $(.,.)$ such that $xv_1, v_2) = (v_1, x^*v_2)$ for all $v_1, v_2 \in V_\varepsilon(\lambda), x \in U(\mathcal{G})$. Let $\{v^\nu_\mu\}$ be an orthogonal basis for weight space $V_\varepsilon(\lambda)_\mu \quad \mu \in P^+$. Then $\bigcup\{v^\nu_\mu\}$ is an orthogonal basis for $V_\varepsilon(\lambda)$. Let $C^\lambda_{\nu,s,\mu,r}(x) = (xv^r_\mu, v^s_\nu)$ be the associated matrix elements of $V_\varepsilon(\lambda)$. Then the matrix elements $C^\lambda_{\nu,s,\mu,r}$(where $\lambda$ runs throngh $P^+$, while $(\mu, r)$ and $(\nu, s)$ runs independently through the index set of a basis of $V_\varepsilon(\lambda)$ form a basis of $\mathcal{F}_\varepsilon(G)$(see [3]).

Now very irreducible $^*$−representation of $\mathcal{F}_\varepsilon(SL_2(\mathbb{C})$ is equivalent to a representation belonging to one of the following two families, each of which is parameterized by $S^1 = \{t \in \mathbb{C}\backslash|t| = 1\}$

i) the family of one-dimensional representations $\mathcal{T}_t$

ii) the family $\pi_t$ of representations in $\ell^2(\mathbb{N})$(see [3]).

Moreover, there exists a surjective homomorphism $\mathcal{F}_\varepsilon(G) \longrightarrow \mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$ induced by the natural inclusion $SL_2\mathbb{C} \hookrightarrow G$ and by composing the representation $\pi_{-1}$ of $\mathcal{F}_\varepsilon(SL_2\mathbb{C})$ with this homomorphic, we obtain a representation of $\mathcal{F}_\varepsilon(G)$ in $\ell^2(\mathbb{N})$ denoted by $\pi_{s_i}$, where $s_i$ appears in the reduced decomposition $\omega = s_{i_1}, s_{i_2}, ..., s_{i_k}$. More precisely, $\pi_{s_i} : \mathcal{F}_\varepsilon(G) \longrightarrow \mathcal{L}(\ell^2(\mathbb{N}))$ is of class CCR(see [11]),i.e, its image is dense in the ideal of compact operators $\mathcal{L}(\ell^2(\mathbb{N}))$.

Then representation $\mathcal{T}_t$ is one-dimensional and is of the form

$$\mathcal{T}_t(C^{\lambda}_{\nu,s,\mu,r}) = \delta_{r,s}\delta_{\mu,\nu}\exp(2\pi\sqrt{-1}\mu(x)),$$

if $t = \exp(2\pi\sqrt{-1}\mu(x) \in \mathbf{T}$, for $x \in$Lie$\mathbf{T}$(see [3]).

**Proposition 3.1.** ([3], 13.1.7). *Every irreducible unitary representation of $\mathcal{F}_\varepsilon(G)$ on a separable Hilbert space is the completion of a unitarizable highest weight representation. Moreover, two such representation are equivalent if and only if they have the same highest weight.*

**Proposition 3.2.** ([3],13.1.9) *Let $\omega = s_{i_1}, s_{i_2}, ..., s_{i_k}$ be a reduced decomposition of an element$\omega$ of the Weyl group $W$ of $G$. Then*

*i) The Hilbert space tensor product $\rho_{\omega,t} = \pi_{s_{i_1}} \otimes \pi_{s_{i_2}} \otimes ...... \otimes \pi_{s_{i_t}} \otimes \mathcal{T}_t$ is an irreducible $^*-$representation of $\mathcal{F}_\varepsilon(G)$ which is associated to the Schubert cell $S_\omega$;*

*ii) Up to equivalence, the representation $\rho_{\omega,t}$ does not depend on the choice of the reduced decomposition of $\omega$;*

*iii) Every irreducible $^*-$representation of $\mathcal{F}_\varepsilon(G)$ is equivalent to some $\rho_{\omega,t}$.*

*The sphere $S^n$, can be realized as the orbit under the action of the compact group $SU(n+1)$ of the highest weight vector $v_0$ in its natural $(n+1)-$dimensional representation $V$ of $SU(n+1)$. If $t_{rs}$, $0 \le r, s \le n$, are the matrix entries of $V$ , the algebra of functions on the orbit is generated by the entries in the "first column" $t_{s0}$ and their complex conjugates. In fact,*

$$\mathcal{F}(S^n) := \mathbb{C}[t_{00}, ..., t_{n0}, \bar{t}_{00}, ..., \bar{t}_{n0}]/\sim,$$

*where " $\sim$ " is the following equivalence relation*

$$t_{s0} \sim \bar{t}_{s0} \Longleftrightarrow \sum_{s=0}^{n} t_{s0}\bar{t}_{s0} = 1.$$

**Proposition 3.3.** ([3], 13.2.6). *The $^*-$structure on Hopf algebra $\mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$, is given by*

$$t^*_{rs} = (-\varepsilon)^{r-s} q \det(\widehat{T}_{rs}),$$

*where $\widehat{T}_{rs}$ is the matrix obtained by removing the $r^{th}$ row and the $s^{th}$ column from $T$.*

**Definition 3.2.** ([3],13.2.7). *The $^*-$subalgebra of $\mathcal{F}_\varepsilon(SL_{n+1}(\mathbb{C}))$ generated by he elements $t_{so}$ and $t*_{s0}$, for $s = 0, ..., n$, is called the quantized algebra of functions on the sphere $S^n$, and is denoted by $\mathcal{F}_\varepsilon(S^n)$. It is a quantum $SL_{n+1}(\mathbb{C})-$space.*

We set $z_s = t_{s0}$ from now on. Using Proposition 2.4, it is easy to see that the following relations hold in $\mathcal{F}_\varepsilon(S^n)$:

$$\begin{cases} z_r.z_s = \varepsilon^{-1} z_s z_r \text{ if } r < s \\ z_r.z^*_s = \varepsilon^{-1} z^*_s z_r \text{ if } r \neq s \\ z_r.z^*_r - z^*_r.z_r + (\varepsilon^{-2}+1)\sum_{s>r} z_s.z^*_s = 0 \\ \sum_{s=0}^{n} z_s.z^*_s = 0. \end{cases} \quad \text{(CP)}$$

Hence,$\mathcal{F}_\varepsilon(S^n)$ has $(CP)$ as its defining relations. The construction of irreducible $^*-$representation of $\mathcal{F}_\varepsilon(S^n)$, is given by.

**Theorem 3.4.** ([3],13.2.9). *Every irreducible $^*-$representation of $\mathcal{F}_\varepsilon(S^n)$ is equivalent exactly to one of the following:*

*i) the one-dimensional representation $\rho_{0,t}$ $t \in S^1$, given by $\rho_{0,t}(z^*_0) = t^{-1}$ and $\rho_{0,t}(z^*_r) = 0$ if $r > 0$,* *ii) the representation $\rho_{0,t}$, $\quad 1 \leq r \leq n, \quad t \in S^1$, on the Hilbert space tensor product $\ell^2(\mathbb{N})^{\otimes r}$, give by*

$$\rho_{r,t}(z_s^*)(e_{k_1} \otimes ... \otimes e_{k_r}) =$$

$$\begin{cases} \varepsilon^{(\sum_{i=1}^{s} k_i + s)}(1 - \varepsilon^{-2(k_{s+1}+1)})^{-2} e_{k_1} \otimes ... \otimes e_{k_s} \otimes e_{k_{s+1}} + 1 \otimes e_{k_{s+2}} \otimes ... \otimes e_{k_s} & \text{if } s < r \\ t^{-1} \varepsilon^{(\sum_{j=1}^{r} k_j + r)} e_{k_1} \otimes ... \otimes e_{k_r} & \text{if } s = r \\ 0 & \text{if } s > r. \end{cases}$$

*The representation $\rho_{0,t}$ is equivalent $t_0$ the restriction of the representation $\mathcal{T}_t$ of $\mathcal{F}_\varepsilon(SL_{n+1}(\mathbb{C}))$ (cf.2.3); and or $r > 0, \rho_{r,t}$ is equivalent to the restriction of $\pi_{s_1} \otimes ... \otimes \pi_{s_r} \otimes \mathcal{T}_t$.*

From Theorem 2.6, we have

$$\bigcap_{(\omega,t) \in W \times T} \ker \rho_{\omega,t} = \{0\},$$

i.e. the representation $\oplus_{\omega \in W} \int_T^{\oplus} \rho_{\omega,t} dt$ is faithful and

$$\dim \rho_{\omega,t} = \begin{cases} 1 \text{ if } \omega = e \\ 0 \text{ if } \omega \neq e. \end{cases}$$

We recall now the definition of compact quantum of spheres $C^*-$ algebra.

**Definition 3.3.** *The $C^*-$algebraic compact quantum sphere $C_\varepsilon^*(S^n)$ is he $C^*-$completion of the $*-$algebra $\mathcal{F}_\varepsilon(S^n)$ with respect to the $C^*-$norm*

$$\|f\| = \sup_\rho \|\rho(f)\|, \quad f \in \mathcal{F}_\varepsilon(S^n)$$

*where $\rho$ runs through the$*-$ representations of $\mathcal{F}_\varepsilon(S^n)$ (cf., Theorem 2.6) and the norm on the right-hand side is the operator.*

It suffcies to show that $\|f\|$ is finite for all $f \in \mathcal{F}_\varepsilon(S^n)$, for it is clear that $\|.\|$ is a $C^*-$norm, i.e. $\|f.f^*\| = \|f\|^2$. We now prove that following result about he structure of compact quantum $C^*-$algebra of sphere $S^n$.

**Theorem 3.5.** *With notation as above, we have*

$$C_\varepsilon^*(S^n) \cong \mathbb{C}(S^1) \oplus \bigoplus_{e \neq \omega \in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{\omega,t}) dt,$$

*where $\mathbb{C}(S^1)$ is the algebra of complex valued continuous functions on $S^1$ and $\mathcal{K}(H)$ ideal of compact operators in a separable Hilbert space $H$.*

*Proof.* Let $\omega = s_{i_1}.s_{i_2}...s_{i_k}$ be a reduced decomposition of the element $\omega \in W$ into a product of reflections. Then by Proposition 2.6, for $r > o$, the representation $\rho_{\omega,t}$ is equivalent to the restriction of $\pi_{s_{i_1}} \otimes .... \otimes \pi_{s_{i_k}} \otimes \mathcal{T}_t$, where $\pi_{s_1}$ is the composition of the homomorphism of $\mathcal{F}_\varepsilon(G)$ onto $\mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$ and the representation $\pi_{-1}$ of $\mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$ in the Hilbert space $\ell^2(\mathbb{N})^{\otimes r}$; and the family of one-dimensional representations $\mathcal{T}_t$, given by

$$\mathcal{T}_t(a) = t, \mathcal{T}_t(b) = \mathcal{T}_t(c) = 0, \mathcal{T}_t(d) = t^{-1},$$

where $t \in S^1$ and $a, b, c, d$ are give by: Algebra$\mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$ is generated by the matrix elements of type $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Hence, by construction the representation $\rho_{\omega,t} = \pi_{s_{i_1}} \otimes .... \otimes \pi_{s_{i_k}} \otimes \mathcal{T}_t$. Thus, we have

$$\pi_{s_i} : C_\varepsilon^*(S^n) \longrightarrow C_\varepsilon^*(SL_2(\mathbb{C})) \xrightarrow{\pi_{-1}} \mathcal{L}(\ell^2(\mathbb{N})^{\otimes r}).$$

Now, $\pi_{s_i}$ is CCR (see, [11]) and so, we have $\pi_{s_i}(C^*_\varepsilon(S^n) \cong \mathcal{K}(H_{\omega,t})$. Moreover $\mathcal{T}_t(C^*_\varepsilon(S^n)) \cong \mathbb{C}$.

Hence,

$$\begin{aligned}\rho_{\omega,t}(C^*_\varepsilon(S^n)) &= (\pi_{s_{i_1}} \otimes ... \otimes \pi_{s_{i_k}} \otimes \mathcal{T}_t)(C^*_\varepsilon(S^n) \\ &= \pi_{s_{i_1}}(C^*_\varepsilon(S^n)) \otimes ... \otimes \pi_{s_{i_k}}(C^*_\varepsilon(S^n)) \otimes \mathcal{T}_t(C^*_\varepsilon(S^n)) \\ &\cong \mathcal{K}(H_{s_{i_1}}) \otimes ... \otimes \mathcal{K}(H_{s_{i_k}}) \otimes \mathbb{C} \\ &\cong \mathcal{K}(H_{\omega,t}),\end{aligned}$$

where $H_{\omega,t} = H_{s_1} \otimes .... \otimes H_{s_i} \otimes \mathbb{C}$.

Thus, $\rho_{\omega,t}(C^*_\varepsilon(S^n)) = \mathcal{K}(H_{\omega,t})$.

Hence,

$$\bigoplus_{\omega \in W} \int_{S^1}^{\oplus} \rho_{\omega,t}(C^*_\varepsilon(S^n)) \cong \bigoplus_{\omega \in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{\omega,t})dt.$$

Now, recall a result of S. Sakai from [11]: Let $A$ be a commutative $C^*-$algebra and $B$ be a $C^*-$algebra. Then, $C_0(\Omega, B) \cong A \otimes B$, where $\Omega$ is the spectrum space of $A$.

Applying this result, for $B = \mathcal{K}(H_{\omega,t}) \cong \mathcal{K}$ and $A = \mathbb{C}(W \times S^1)$ be a commutative $C^*-$algebra. Thus, we have

$$C^*_\varepsilon(S^n) \cong \mathbb{C}(S^1) \oplus \bigoplus_{e \neq \omega \in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{\omega,t})dt.$$

Now, we first compute the $K_*(C^*_\varepsilon(S^n))$ and the $HE_*(C^*_\varepsilon(S^n))$ of $C^*-$algebra of quantum sphere $S^n$.

**Proposition 3.6.**

$$HE_*(C^*_\varepsilon(S^n)) \cong H^*_{DR}(W \times S^1).$$

*Proof.* We have

$$\begin{aligned}HE_*(C^*_\varepsilon(S^n)) &= HE_*(\mathbb{C}(S^1) \oplus \bigoplus_{e \neq \omega \in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{\omega,t})dt) \\ &= HE_*(\mathbb{C}(S^1) \oplus HE_*( \bigoplus_{e \neq \omega \in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{\omega,t})dt)) \\ &\cong HE_*(\mathbb{C}(W \times S^1) \otimes \mathcal{K} \quad \text{(by Proposition 1.1)} \\ &\cong HE_*(\mathbb{C}(W \times S^1)).\end{aligned}$$

Since $C(W \times S^1)$ is a commutative $^*-$algebra, by Proposition 1.5 §1, we have

$$HE_*(C^*_\varepsilon(S^n)) \cong HE_*(\mathbb{C}(W \times S^1)) \cong H^*_{DR}(W \times S^1)).$$

**Proposition 3.7.**

$$K_*(C^*_\varepsilon(S^n)) \cong K^*(W \times S^1).$$

*Proof.* We have

$$\begin{aligned} K_*(C^*_\varepsilon(S^n)) &= K_*(\mathbb{C}(S^1) \oplus \bigoplus_{e\neq\omega\in W} \int^{\oplus}_{S^1} \mathcal{K}(H_{\omega,t})dt) \\ &= K_*(\mathbb{C}(S^1) \oplus K_*( \bigoplus_{e\neq\omega\in W} \int^{\oplus}_{S^1} \mathcal{K}(H_{\omega,t})dt)) \\ &\cong K_*(\mathbb{C}(W \times S^1) \otimes \mathcal{K} \quad \text{(by Proposition 1.1)} \\ &\cong K_*(\mathbb{C}(W \times S^1)). \end{aligned}$$

In result of Proposition 1.5, §1, we have

$$K_*(\mathbb{C}(W \times S^1)) \cong K_*(W \times S^1).$$

**Theorem 3.8.** *With notation above, the Chern character of $C*-$algebra of quantum sphere $C*_\varepsilon(S^n)$*

$$ch_{C^*} : K_*(C*_\varepsilon(S^n) \longrightarrow HE_*(C*_\varepsilon(S^n))$$

*is an isomorphism.*

*Proof.* By Proposition 2.9 and 2.10, we have

$$HE_*(C^*_\varepsilon(S^n)) \cong HE_*(\mathbb{C}(W \times S^1)) \cong H^*_{DR}(W \times S^1)),$$

$$K_*(C^*_\varepsilon(S^n)) \cong K_*(\mathbb{C}(W \times S^1)) \cong K^*(W \times S^1)).$$

Now, consider the commutative diagram

$$\begin{array}{ccc} K_*(C^*_\varepsilon(S^n)) & \xrightarrow{ch_{C^*_\varepsilon}} & HE_*(C^*_\varepsilon(S^n)) \\ \cong\downarrow & & \downarrow\cong \\ K_*(\mathbb{C}(W \times S^1)) & \xrightarrow{ch_{CQ}} & HE_*(\mathbb{C}(W \times S^1)) \\ \cong\downarrow & & \downarrow\cong \\ K^*(W \times S^1) & \xrightarrow{ch} & H^*_{DR}(W \times S^1)). \end{array}$$

Moreover, follwing Watanabe [15], the $ch : K^*(W \times S^1) \otimes \mathbb{C} \longrightarrow H^*_{DR}(W \times S^1)$ is an isomorphism.

Thus, $ch_{C^*} : K_*(C*_\varepsilon(S^n) \longrightarrow HE_*(C*_\varepsilon(S^n))$ is an isomorphism.

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