VNU Journal of Science, Mathematics - Physics 25 (2009) 237-248

Series representation of random mappings and their extension

Dang Hung Thang*, Tran Manh Cuong

Department of Mathematics, Vietnam National University, 334 NguyenTrai Str., Hanoi, Vietnam

Received 28 February 2009

Abstract. In this paper, we introduce a method of extending the domain of a random mapping admitting the series expansion. This method is based on the convergence of certain random series. Some conditions under which a random mapping can be extended to apply to all X - valued random variables will be presented.
AMS Subject classification 2000: Primary $60H05$; Secondary: $60B11$, $60G57$, $60K37$, $37L55$.
Keywords and phrases : random operator, bounded random operator, domain of extension, action on random inputs.

1. Series representation of random mappings

Let X, Y be separable metric spaces. By a random mapping from X into Y we mean a rule Φ that assigns to each element $x \in X$ a unique Y - valued random variable Φx. Equivalently, it is a mapping $\Phi : \Omega \times X \to Y$ such that for each fixed $x \in X$, the map $\Phi(., x) : \Omega \to Y$ is measurable.

In this point of view, two mappings $\Phi_1 : \Omega \times X \to Y$, $\Phi_2 : \Omega \times X \to Y$ define the same random mapping if for each $x \in X$

$$\Phi_1(x, \omega) = \Phi_2(x, \omega) \quad \text{a.s.}$$

Noting that the exceptional set can depend on x. In this case, we say that the random mapping Φ_2 is a modification of the random mapping Φ_1.

Definition 1.1 A random mapping Φ from X into Y is said to admit the series expansion if there exists a sequence (f_n) of deterministic measurable mappings from X into Y (rep. from X into R) and a sequence (α_n) of real-valued random variables (rep. Y-valued r.v.'s) such that

$$\Phi x = \sum_{n=1}^{\infty} \alpha_n f_n x,$$

where the series converges in L_0^Y.

In the case the sequence (α_n) are independent we say that Φ admits an independent series expansion.

* Corresponding authors. Tel. +844.38581135:
E-mail: hungthang.dang@gmail.com

Proposition 1.2 *Let Φ be a random operator from X into Y and suppose that X is a Banach space with the Shauder basis $e = (e_n)_{n=1}^{\infty}$ and the conjugate basis $e^* = (e_n^*)_{n=1}^{\infty}$. Then Φ admits the series expansion.*

Recall that, a random mapping Φ is called a random operator if it is linear and stochastically continuous, i.e.

$$\Phi(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 \Phi x_1 + \lambda_2 \Phi x_2, \ a.s. \ \forall x_1, x_2 \in X, \lambda_1, \lambda_2 \in \mathbb{R},$$

and

$$\text{p-lim}_{x \to x_0} \Phi x = \Phi x_0.$$

Note that the exceptional set may depend on $\lambda_1, \lambda_2, x_1, x_2$.

Proof. For each $x \in X$, we have

$$x = \sum_{n=1}^{\infty} (x, e_n^*) e_n.$$

Since Φ is linear and stochastically continuous, we get

$$\Phi x = \sum_{n=1}^{\infty} (x, e_n^*) \Phi e_n$$

where the series converges in L_0^Y.

Put $\alpha_n = \Phi e_n, f_n(x) = (x, e_n^*)$. (α_n) is a sequence of Y-valued and (f_n) of deterministic measurable mappings from X into Y. We have

$$\Phi x = \sum_{n=1}^{\infty} \alpha_n f_n x.$$

□

A random mapping Φ from X into Y is called a symmetric Gaussian random mapping if for each $k \in \mathbb{N}$ and for each finite sequence $(x_i, y_i^*)_{i=1}^k$ of $X \times Y^*$ the $\mathbb{R}^k$ - valued random variable $\{(\Phi x_i, y^*)\}_{i=1}^k$ is symmetric and Gaussian.

Theorem 1.3 *Let Φ be a symmetric stochastically continuous Gaussian random mapping. Then Φ admits an independent series expansion*

$$\Phi x = \sum_{n=1}^{\infty} \alpha_n f_n x,$$

where (α_n) is a sequence of real-valued Gaussian i.i.d random variables and $f_n : X \to Y$ is continuous (so is measurable).

Proof. Let $[\Phi]$ denote the closed subspace of $L_2(\Omega)$ spanned by random variables $\{(\Phi x, y^*), x \in X, y^* \in Y^*\}$. Then $[\Phi]$ is a separable Hilbert space and every element of $[\Phi]$ is a symmetric Gaussian random variable. Let (α_n) is an orthonormal basis of $[\Phi]$. Since the sequence (α_n) is orthogonal, symmetric and Gaussian, it is a sequence of real-valued Gaussian i.i.d random variables. Now for each n, we define a mapping $f_n : X \to Y$ by

$$f_n x = \int_{\Omega} \alpha_n(\omega) \Phi x(\omega) dP(\omega). \tag{1}$$

Here the Bochner (1) exists because by Cauchy inequality

$$\int_\Omega \|\alpha_n(\omega)\Phi x(\omega)\| dP(\omega) \le \{E\|\Phi x\|^2\}^{1/2}. \tag{2}$$

Fix $x \in X$. For each $y^* \in Y^*$, $(\Phi x, y^*) \in [\Phi]$ is expanded in the basis (α_n) in the form

$$\begin{aligned}(\Phi x, y^*) &= \sum_{n=1}^{\infty}\left(\int_\Omega (\Phi x, y^*)\alpha_n dP(\omega)\right)\alpha_n \\ &= \sum_{n=1}^{\infty}\left(\int_\Omega \alpha_n \Phi x dP(\omega), y^*\right)\alpha_n \\ &= \sum_{n=1}^{\infty}(\alpha_n f_n x, y^*)\end{aligned}$$

where the series converges in $L_2(\Omega)$ so it is convergent in probability. Since the sequence $(\alpha_n f_n x)$ is a sequence of symmetric independent Y - valued r.v.'s, by the Ito - Nisio theorem, we conclude that

$$\Phi x = \sum_{n=1}^{\infty}\alpha_n f_n x \quad \text{a.s.}$$

Finally, fixing n, we show that f_n is continuous. Let $(x_k) \subset X$ such that $\lim_k x_k = x$. From (2) we have

$$\|f_n x_k - f_n x\|^2 \le E\|\Phi x_k - \Phi x\|^2.$$

By the assumption p$-\lim \Phi x_k = \Phi x$ and the fact that in $[\Phi]$ all the convergence in $L_p(\Omega), (p \ge 0)$ are equivalent, we have $\lim_k E\|\Phi x_k - \Phi x\|^2 = 0$. Therefore, $\lim_k f_n x_k = f_n x$. □

Next, we shall be interested in possible extensions of Theorem 1.3 to the case of symmetric stable random mappings.

Let Φ be a random mapping from X into Y. Φ is said to be a symmetric p-stable random mapping (SpS random mapping in short) if the real process $\{(\Phi x, y^*)\}$ defined on $X \times Y^*$ is symmetric p - stable. In this case, for each $x \in X$, Φx is a Y-valued SpS random variable.

Let $[\Phi]$ denote the closed subspace of $L_0(\Omega)$ spanned by random variables $\{(\Phi x, y^*), x \in X, y^* \in Y^*\}$. If $\xi \in [\Phi]$ then ξ is SpS so the ch.f. of ξ is of the form $\exp\{-c|t|^p\}$, where $c = c(\xi)$ is a non-negative number depending on ξ. The length of ξ denoted by $\|\xi\|_*$ is defined by

$$\|\xi\|_* = \{c(\xi)\}^{1/p}.$$

It is known that (see [1]).

Lemma

i) The correspondence $\xi \mapsto \|\xi\|_*$ is an F-norm on $[\Phi]$ and in fact is a norm in the case $p \ge 1$.

ii) $[\Phi]$ is a linear subspace of each $L_r(\Omega), 0 \le r < p$ and all topologies $L_r(\Omega)$, $0 \le r < p$ coincide with the topology induced by $\|\xi\|_*$ - norm on $[\Phi]$.

iii) The F - space $[\Phi]$ can be isometrically embedded into some $L_p(S, \mathcal{A}, \mu)$.

Theorem 1.4 *Let Φ be SpS stochastically continuous random mapping and suppose that $[\Phi]$ is isometric to l_p. Then Φ admits an independent series expansion*

$$\Phi x = \sum_{n=1}^{\infty}\alpha_n f_n x,$$

where (α_n) is a sequence of real-valued SpS i.i.d random variables and $f_n : X \to Y$ is continuous (so it is measurable).

Proof. Let I be an isometry from $[\Phi]$ onto l_p and $J = I^{-1}$. Put

$$\alpha_n = J(e_n),$$
$$I\left((\Phi x, y^*)\right) = B(x, y^*) \in l_p.$$

At first, we shall show that (α_n) is a sequence of real-valued SpS i.i.d random variables. Indeed, the joint ch.f. $f(t_1, t_2, ..., t_n)$ of the random variable $(\alpha_1, \alpha_2, ..., \alpha_n)$ is equal to

$$\begin{aligned} f(t_1, t_2, ..., t_n) &= E\exp\left\{i\sum_{k=1}^{n} t_k\alpha_k\right\} = E\exp\left\{i\sum_{k=1}^{n} t_kJ(e_k)\right\} \\ &= E\exp\left\{iJ\left(\sum_{k=1}^{n} t_ke_k\right)\right\} = \exp\left\{-\left\|J\left(\sum_{k=1}^{n} t_ke_k\right)\right\|_*^p\right\} \\ &= \exp\left\{-\left\|\sum_{k=1}^{n} t_ke_k\right\|^p\right\} = \exp\left\{-\sum_{k=1}^{n} |t_k|^p\right\} \end{aligned}$$

as desired.

For each $(x, y^*) \in X \times Y^*$, we have

$$\alpha_n = J(e_n),$$
$$I((\Phi x, y^*)) = B(x, y^*) \in l_p,$$

hence

$$(\Phi x, y^*) = \sum_{n=1}^{\infty} \alpha_n b_n(x, y^*), \tag{3}$$

where $b_n(x, y^*)$ is the n-th coordinate of $B(x, y^*) \in l_p$ and the series (3) converges in the norm $\|.\|_*$ so converges in probability.

Fix n. We show that there exists a mapping $f_n : X \to Y$ such that for each $x \in X, y^* \in Y^*$

$$b_n(x, y^*) = (f_nx, y^*).$$

Fix $x \in X$. Since the mapping $y^* \mapsto (\Phi x, y^*)$ is linear so the mapping $y^* \mapsto B(x, y^*)$ is linear which implies the mapping $b_x : y^* \mapsto b_n(x, y^*)$ from Y^* into R is linear. In addition, the ch.f. of Φx is $\tau(Y^*, Y)$- continuous on Y^*, where $\tau(Y^*, Y)$ is the topology of uniform convergence on compact sets of Y, and it is equal to

$$H_x(y^*) = \exp\{-\|(\Phi x, y^*)\|_*^p\} = \exp\{-\|B(x, y^*)\|^p\}$$

Consequently, $b_x : Y^* \to R$ is linear and $\tau(Y^*, Y)$- continuous on Y^*. Since the dual space of Y^* under the topology $\tau(Y^*, Y)$ is Y we conclude that there exists a unique element denoted by f_nx such that

$$b_x(y^*) = (f_nx, y^*) \to b_n(x, y^*) = (f_nx, y^*).$$

Now, the equality (3) becomes

$$(\Phi x, y^*) = \sum_{n=1}^{\infty} \alpha_n b_n (x, y^*)$$
$$= \sum_{n=1}^{\infty} (\alpha_n f_n x, y^*).$$

The rest of proof is carried out similarly as in the proof of Theorem 1.3.

Finally, fixing n, we show that f_n is continuous. Let (x_k) be a sequence of X such that $\lim_k x_k = x$. By the assumption p-lim $\Phi x_k = \Phi x$, we have

$$\Phi x_k - \Phi x = \sum_{j=1}^{\infty} \alpha_j (f_j x_k - f_j x).$$

Since $p < 2$ by Corrolary 7.3.6 in [2], we get

$$\|f_n x_k - f_n x\|^p \le \sum_{j=1}^{\infty} \|f_j x_k - f_j x\|^p \le C\{E\|\Phi x_k - \Phi x\|^r\}^{p/r},$$

where $r < p$ and the constant $C > 0$ depends only on r, p. From 2. of Lemma we obtain $\lim_k \{E\|\Phi x_k - \Phi x\|^r\}^{1/r} = 0$. Hence, $\lim_k f_n x_k = f_n x$ as desired. □

2. The extension of random mappings admitting series expansion

Let Φ be a random mapping from X into Y admitting the series expansion

$$\Phi x = \sum_{n=1}^{\infty} \alpha_n f_n x, \tag{4}$$

where (f_n) is a sequence of deterministic measurable mappings from X into Y (rep. from X into R) and (α_n) is a sequence of real - valued random variable (rep. Y - valued r.v.). The series converges in L_0^Y.

Denote by $\mathcal{D}(\Phi)$ the set of all X - valued r.v. u such that the series

$$\sum_{n=1}^{\infty} \alpha_n f_n u \tag{5}$$

converges in probability. Here $f_n u(\omega) = f_n(u(\omega))$ is a random variable because f_n is measurable.

Clearly, $X \subset \mathcal{D}(\Phi) \subset L_0^X$.

Definition 2.1 $\mathcal{D}(\Phi)$ is called the domain of extension of Φ. If $u \in \mathcal{D}(\Phi)$ then the sum (5) is denoted by Φu and it is understood as the action of Φ on the random variable u.

Theorem 2.2 *If u is a countably - valued r.v.*

$$u = \sum_{i=1}^{\infty} 1_{E_i} x_i,$$

where $(E_i, i = 1, 2, ...)$ *is a countable partition of* Ω *and* $x_i \in X$, *then* $u \in \mathcal{D}(\Phi)$ *and*

$$\Phi u = \sum_{i=1}^{\infty} 1_{E_i} \Phi x_i.$$

Proof. Put $Z_n = \sum_{i=1}^{n} \alpha_i f_i u$ and $Z = \sum_{i=1}^{\infty} 1_{E_i} \Phi x_i$. We have to show that

$$\lim_n P(\|Z_n - Z\| > t) = 0.$$

Since $\omega \in E_k \Rightarrow Z(\omega) = \Phi x_k, Z_n(\omega) = \sum\limits_{i=1}^{n} \alpha_i f_i x_k$ so

$$\begin{aligned} P(\|Z_n - Z\| > t) &= \sum_{k=1}^{\infty} P(\|Z_n - Z\| > t, E_k) \\ &\leq \sum_{k=1}^{N} P\left(\left\|\sum_{i=1}^{n} \alpha_i f_i x_k - \Phi x_k\right\| > t\right) + \sum_{k=N+1}^{\infty} P(E_k) \end{aligned}$$

For each $k = 1, 2, .., N$ we have

$$\lim_n P(\|\sum_{i=1}^{n} \alpha_i f_i x_k - \Phi x_k\| > t) = 0.$$

Let $n \to \infty$ and then $N \to \infty$, we get $\lim_n P(\|Z_n - Z\| > t) = 0$. $\square$

For each random mapping Φ admitting the representation (4), let $\mathcal{F}(\alpha)$ denote the σ-algebra generated by the family $\{\alpha_n\}$. A random variable $u \in L_0^X$ is said to be independent of Φ if $\mathcal{F}(u)$ and $\mathcal{F}(\alpha)$ are independent.

Theorem 2.3 *Suppose that* u *is independent of* Φ, *then* $u \in \mathcal{D}(\Phi)$.

Proof. Let $t > 0$. By the independence of u and the sequence (α_n) we have

$$P\left(\left\|\sum_{i=m}^{n} \alpha_i f_i u\right\| > t\right) = \int_X P\left(\left\|\sum_{i=m}^{n} \alpha_i f_i x)\right\| > t\right) d\mu(x),$$

where μ is the distribution of u. Because for each $x \in X$

$$\lim_{m,n\to\infty} P\left(\|\sum_{i=m}^{n} \alpha_i f_i x\| > t\right) = 0.$$

By the dominated convergence theorem, we infer that

$$\lim_{m,n\to\infty} P\left(\|\sum_{i=m}^{n} \alpha_i f_i u\| > t\right) = 0.$$

Therefore, the series

$$\sum_{i=1}^{\infty} \alpha_i f_i u$$

converges in L_0^Y i.e. $u \in \mathcal{D}(\Phi)$. $\square$

Theorem 2.4 *Let* Φ *be a random mapping from* X *into* Y *admitting the series expansion of the form (4). Suppose that* $E|\alpha_k|^p < C$ *for all* k, *where* $p > 1$ *and* q *is the conjugate number of* p *(i.e.*

$1/p + 1/q = 1$). For $u \in L_0^X$ to belong to $\mathcal{D}(\Phi)$, a sufficient condition is

$$\sum_k \{E\|f_k u\|^q\}^{1/q} < \infty. \tag{6}$$

Proof. Put

$$r_k(q) = \{E\|f_k u\|^q\}^{1/q}.$$

Applying the Hölder inequality, we get

$$\begin{aligned} E\left\|\sum_{k=m}^{n} \alpha_k f_k u\right\| &\leq \sum_{k=m}^{n} E|\alpha_k|\|f_k u\| \\ &\leq \sum_{k=m}^{n} \{E|\alpha_k|^p\}^{1/p}\{E\|f_k u\|^q\}^{1/q} \\ &\leq C\sum_{k=m}^{n} r_k(q) \to 0 \quad \text{as } m, n \to \infty. \end{aligned}$$

Hence, the series $\sum\limits_{k=1}^{\infty} \alpha_k f_k u$ converges in L_1^Y so converges in L_0^Y. □

Corrolary 2.5 *Suppose that Φ is a symmetric stochastically continuous Gaussian random mapping and if*

$$\sum_k \{E\|(f_k u)\|^q\}^{1/q} < \infty$$

for some $q > 1$ then $u \in \mathcal{D}(\Phi)$.

3. When a random mapping can be extended to the entire space L_0^X

Let Φ be a random operator from X into Y and suppose that X is a separable Banach space with the Shauder basis $e = (e_n)_{n=1}^{\infty}$ and the conjugate basis $e^* = (e_n^*)_{n=1}^{\infty}$. By Proposition 1.2, Φ admits the series expansion.

$$\Phi x = \sum_{n=1}^{\infty} (x, e_n^*)\Phi e_n.$$

Theorem 3.1

i) If Φ is a bounded random operator then $\mathcal{D}(\Phi) = L_0^X$ and Φu does not depend on the basis (e_n).

ii) Conversely, if $\mathcal{D}(\Phi) = L_0^X$ then Φ must be a bounded random operator.

Recall that (see[3]) a random operator Φ is said to be bounded if there exists a real-valued random variable $k(\omega)$ such that for each $x \in X$

$$\|\Phi x(\omega)\| \leqslant k(\omega)\|x\| \quad a.s.$$

Noting that the exceptional set may depend on x.

Proof: i) Since Φ is bounded, by Theorem 3.1 [3] there exists a mapping

$$T : \Omega \to L(X, Y)$$

such that for each $x \in X$

$$\Phi x(\omega) = T(\omega)x \qquad \text{a.s.}$$

As a consequence, there is a set D with $\mathbb{P}(D) = 1$ such that for each $\omega \in D$ and for all n we have

$$\Phi e_n(\omega) = T(\omega)e_n.$$

Hence, for each $\omega \in D$

$$\sum_{n=1}^{\infty}(u(\omega), e_n^*)\Phi e_n(\omega) = \sum_{n=1}^{\infty}(u(\omega), e_n^*)T(\omega)e_n$$
$$= T(\omega)\left(\sum_{n=1}^{\infty}(u(\omega), e_n^*)e_n\right) = T(\omega)(u(\omega)).$$

Therefore, the series $\sum_{n=1}^{\infty}(u, e_n^*)\Phi e_n$ converges a.s. so converges in probability. Consequently, $u \in \mathcal{D}(\Phi)$ and $\Phi u(\omega) = T(\omega)(u(\omega))$ does not depend on the basis $e = (e_n)$.

ii) Put

$$\Phi_n u = \sum_{i=1}^{n}(u, e_n^*)\Phi e_n.$$

Then Φ_n is a linear continuous mapping from L_0^X into L_0^Y. By the assumption $\lim_n \Phi_n u = \Phi u$ for all $u \in L_0^X$. Hence, by the Banach-Steinhaus theorem Φ is again a linear continuous mapping from L_0^X into L_0^Y. In addition, we have

$$\Phi(u) = \sum_{i=1}^{n} 1_{E_i}\Phi x_i$$

for $u = \sum_{i=1}^{n} 1_{E_i}x_i$ where $(E_i, i = 1, ..., n)$ is a partition of Ω and $x_i \in X$. By Theorem 5.3 [3] we conclude that Φ is bounded. $\square$

Theorem 3.2 *Let Φ be a random operator admitting the series expansion of the form (4), where (α_n) is a sequence of real-valued random variables and (f_n) is a sequence of continuous linear mappings from X into Y. Then*

i) If Φ is bounded then $\mathcal{D}(\Phi) = L_0^X$.

ii) Conversely, if $\mathcal{D}(\Phi) = L_0^X$ then Φ must be bounded.

Proof:i) Since Φ is bounded, by Theorem 3.1 [3] there exists a mapping

$$T : \Omega \to L(X, Y)$$

such that for each $x \in X$

$$\Phi x(\omega) = T(\omega)x \qquad \text{a.s.}$$

For this reason, there is a set D with $\mathbb{P}(D) = 1$ such that for each $\omega \in D$ and for all k we have

$$\Phi e_k(\omega) = \sum_n \alpha_n(\omega)f_n e_k = T(\omega)e_k.$$

As a consequence, for each $\omega \in D$

$$\begin{aligned}\sum_n \alpha_n(\omega) f_n u(\omega) &= \sum_n \alpha_n(\omega) f_n(\sum_k < u(\omega), e_k^* > e_k) \\ &= \sum_n \alpha_n(\omega) \sum_k < u(\omega), e_k^* > f_n e_k \\ &= \sum_k < u(\omega), e_k^* > \sum_n \alpha_n(\omega) f_n e_k \\ &= \sum_k < u(\omega), e_k^* > T(\omega) e_k \\ &= T(\omega)\left(\sum_k < u(\omega), e_k^* > e_k\right) \\ &= T(\omega)(u(\omega)).\end{aligned}$$

ii) Put

$$\Phi_n u = \sum_{i=1}^{n} \alpha_i f_i u.$$

Then Φ_n is a linear continuous mapping from L_0^X into L_0^Y. By the assumption $\lim_n \Phi_n u = \Phi u$ for all $u \in L_0^X$. Hence, by the Banach - Steinhaus theorem Φ is again a linear continuous mapping from L_0^X into L_0^Y. In addition, for $u = \sum_{i=1}^n 1_{E_i} x_i$ where $(E_i, i = 1, ..., n)$ is a partition of Ω and $x_i \in X$, we have

$$\begin{aligned}\Phi(u) &= \sum_{k=1}^{\infty} \alpha_k f_k u \\ &= \sum_{k=1}^{\infty} \alpha_k \sum_{i=1}^{n} 1_{E_i} f_k x_i = \sum_{i=1}^{n} 1_{E_i} \sum_{k=1}^{\infty} \alpha_k f_k x_i \\ &= \sum_{i=1}^{n} 1_{E_i} \Phi x_i.\end{aligned}$$

By Theorem 5.3 [3] we conclude that Φ is bounded. □

Theorem 3.3 *Let X be a compact metric space and Φ be a random mapping from X into Y admiting the series expansion of the form (4), where (α_n) is a sequence of real-valued symmetric independent random variables and (f_n) is a sequence of continuous mappings from X into Y.*

i) If Φ has a continuous modification then every $u \in L_0^X$ belongs to $\mathcal{D}(\Phi)$ i.e. $\mathcal{D}(\Phi) = L_0^X$.

ii) The converse is not true i.e. there exists a random mapping Φ from X into Y admiting the series expansion of the form (4), where (α_n) is a sequence of real-valued symmetric independent random variables and (f_n) is a sequence of continuous mappings from X into Y such that $\mathcal{D}(\Phi) = L_0^X$ but Φ has not a continuous modification.

Proof. Let $V = C(X, Y)$ be the set of all continuous mappings from X into Y. It is known that V is a separable Banach space under the supremum norm

$$\|f\|_V = \sup_{x \in X} \|f(x)\|.$$

For each pair $(x, y^*) \in X \times Y^*$ the mapping $x \otimes y^* : V \to \mathbb{R}$ given by

$$(x \otimes y^*)(f) = (f(x), y^*)$$

is clearly an element of V^*. Let $\Gamma = \{(x \otimes y^*), (x, y^*) \in X \times Y^*\}$. It is easy to check that Γ is a separating subset of V^*. Let $\Psi(x, \omega)$ be a continuous modification of Φ. Define a mapping $T : \Omega \to V$ by

$$T(\omega) = x \mapsto \Psi(x, \omega).$$

We show that T is measurable i.e T is a V-valued random variable. Indeed, for each $(x \otimes y^*) \in \Gamma$ the mapping $\omega \mapsto (T(\omega), x \otimes y^*) = (T(\omega)x, y^*) = (\Psi(x, \omega), y^*) = (\Phi x(\omega), y^*)$ a.s. is measurable. Since V is separable and Γ is a separating subset of V^*, the claims follows from the theorem 1.1 in ([4]).

Note that for each ω the mapping $x \mapsto \alpha_n(\omega) f_n x$ is an element of V. Hence $\alpha_n f_n$ is a V-valued r.v. Now for each $(x \otimes y^*) \in \Gamma$ we have

$$\begin{aligned}(T(\omega), x \otimes y^*) &= (T(\omega)x, y^*) = (\Phi x(\omega), y^*) \\ &= \sum_{n=1}^{\infty} (\alpha_n(\omega) f_n x, y^*) = \sum_{n=1}^{\infty} (\alpha_n(\omega) f_n, x \otimes y^*) \quad \text{a.s.}\end{aligned}$$

Since $(\alpha_n f_n)$ is a sequence of V-valued symmetric independent r.v.'s in view of Ito - Nisio theorem we conclude that the series $\sum\limits_{n=1}^{\infty} \alpha_n(\omega) f_n$ converges a.s. to T in the norm of V. This implies that there exists a set D of probability one such that for each $\omega \in D, x \in X$, we have

$$T(\omega)x = \sum_{n=1}^{\infty} \alpha_n(\omega) f_n x.$$

Consequently, for $u \in L_0^X$ we have

$$T(\omega)(u(\omega)) = \sum_{n=1}^{\infty} \alpha_n(\omega) f_n(u(\omega) = \sum_{n=1}^{\infty} \alpha_n(\omega) f_n u(\omega) \quad \forall \omega \in D$$

i.e. the series $\sum_{n=1}^{\infty} \alpha_n(\omega) f_n u(\omega)$ converges a.s.

ii)The following example shows that the converse is not true.

Example. Let $X = [0; 1], Y = \mathbb{R}$. Consider the sequence (ξ_n) of real-valued independent r.v.'s given by

$$P(\xi_n = -n) = P(\xi_n = n) = \frac{1}{2n^2}, P(\xi_n = 0) = 1 - \frac{1}{n^2}.$$

Then (ξ_n) are real-valued symmetric independent r.v.'s and

$$\mathbb{E}(\xi_n) = 0, \mathbb{E}|\xi_n| = \frac{1}{n}, \mathbb{E}|\xi_n|^2 = 1.$$

Let (a_n) be sequence of positive numbers defined by

$$a_n = \frac{1}{\sqrt{n} \log_2 n}$$

and put $\alpha_n = a_n \xi_n$. Then (α_n) are real-valued symmetric independent r.v.'s and

$$\mathbb{E}(\alpha_n) = 0, \mathbb{E}|\alpha_n| = \frac{a_n}{n}, \mathbb{E}|\alpha_n|^2 = a_n^2.$$

Let (f_n) be the sequence of functions $f_n : [0;1] \to \mathbb{R}$ defined by

$$f_n(t) = \cos 2\pi nt.$$

Clearly, f_n are continuous. Consider the random function $\Phi : [0;1] \to \mathbb{R}$ given by

$$\Phi(t)(\omega) = \sum_{n=1}^{\infty} \alpha_n(\omega) f_n(t). \tag{7}$$

We have

$$\sum_{n=1}^{\infty} \mathbb{E}|\alpha_n f_n(t)| \leq \sum_{n=1}^{\infty} \mathbb{E}|\alpha_n| = \sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$$

since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{\ln 4}{n \ln^2 n} < \infty$. This implies the series (7) converges a.s. Moreover, for each real - valued random variable u we have

$$\sum_{n=1}^{\infty} \mathbb{E}|\alpha_n f_n(u)| \leq \sum_{n=1}^{\infty} \mathbb{E}|\alpha_n| = \sum_{n=1}^{\infty} \frac{a_n}{n} < \infty.$$

This implies the series

$$\sum_{n=1}^{\infty} \alpha_n f_n(u)$$

converges a.s. Hence $\mathcal{D}(\Phi) = L_0(\mathbb{R})$.

Next, we shall show that $\Phi(t)$ is an unbounded function. To this end, we use the following result from ([5]) (Theorem 7 and Exersise 3 p. 231).

Consider the random series

$$\Phi(t)(\omega) = \sum_{n=1}^{\infty} a_n \xi_n(\omega) f_n(t), t \in [0;1]$$

where (ξ_n) are independent and symmetric r.v.'s with $\mathbb{E}|\xi_n|^2 = 1$, (a_n) are positive real numbers such that $\sum_n a_n^2 < \infty$ and $f_n(t) = \cos 2\pi nt$. Put

$$s_i = \left(\sum_{2^i \leq n < 2^{i+1}} a_n^2 \right)^{1/2}.$$

Then if $\sum_{i=0}^{\infty} s_i = \infty$ then $\Phi(t)(\omega)$ is not a bounded function on $[0;1]$ a.s.

Now we come back to our example. We have

$$s_i = \left(\sum_{2^i \leq n < 2^{i+1}} a_n^2 \right)^{1/2} \geq \left(2^i a_{2^{i+1}}^2\right)^{1/2} = \frac{1}{\sqrt{2}(i+1)}$$

which implies that $\sum_{i=0}^{\infty} s_i = \infty$. Therefore, for almost sure ω, $\Phi(t)(\omega)$ is not bounded a.s. so is not continuous on $[0;1]$ a.s.

Acknowledgment. This work was supported in part by the Vietnam National Foundation for Science and Technology Development.

References

[1] Bretagnolle, J., Dacunha - Castelle, D. and Krivine, J. L. (1966). Lois stable et espace L_p, *Ann. Inst. H. Poincaré*, B2, 231 - 259.
[2] W.Linde, *Infinitely divisible and stable measures on Banach spaces*, Leipzig 1983.
[3] D.H.Thang and N.Thinh, Random bounded operators and their extension, *Kyushu J. Math.* Vol.58(2004), 257-276.
[4] N.N. Vakhania, V.I. Tarieladze and S.A. Chobanian, *Probability Distribution on Banach spaces*, D.Reidel Publishing Company, Dordrecht. 1987.
[5] J. P. Kahane, *Some random series of functions*, Cambridge Univ. Press 1985.