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A Unifying Statistical Model for Atmospheric Optical Scintillation

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1. Introduction

Atmospheric optical communication has been receiving considerable attention recently for use in high data rate wireless links (Juarez et al., 2006)-(Zhu & Kahn, 2002). Considering their narrow beamwidths and lack of licensing requirements as compared to microwave systems, atmospheric optical systems are appropriate candidates for secure, high data rate, cost-effective, wide bandwidth communications. Furthermore, atmospheric free space optical (FSO) communications are less susceptible to the radio interference than radio-wireless communications. Thus, FSO communication systems represent a promising alternative to solve the last mile problem, above all in densely populated urban areas.

However, even in clear sky conditions, wireless optical links may experience fading due to the turbulent atmosphere. In this respect, inhomogeneities in the temperature and pressure of the atmosphere lead to variations of the refractive index along the transmission path. These random refractive index variations can lead to power losses at the receiver and eventually to fluctuations in both the intensity and the phase of an optical wave propagating through this medium (Andrews & Phillips, 1998). Such fluctuations can produce an increase in the link error probability limiting the performance of communication systems. In this particular scenario, the turbulence-induced fading is called scintillation.

The reliability of an optical system operating in an environment as the mentioned above can be deduced from a mathematical model for the probability density function (pdf) of the randomly fading irradiance signal. For that reason, one of the goals in studying optical wave propagation through turbulence is the identification of a tractable pdf of the irradiance under all irradiance fluctuation regimes.

The purpose of this chapter is to develop a new tractable pdf model for the irradiance fluctuations of an unbounded optical wavefront (plane and spherical waves) propagating through a homogeneous, isotropic turbulence to explain the focusing and strong turbulence regimes where multiple scattering effects are important. Hence, the desired theoretical solution can be useful in studying the performance characteristics of any optical communication system operating through a turbulent atmosphere. We demonstrate through this chapter that our proposed model fits very well to the published data in the literature, and it generalizes in a closed-form expression most of the developed pdf models that have been proposed by the scientific community for more than four decades.

2. Background: distribution models

2.1 Limiting cases of weak turbulence and far into saturation regime.

Over the years, many irradiance pdf models have been proposed with varying degrees of success. Under weak irradiance fluctuations it has been well established that the Born and Rytov perturbation methods (Andrews & Phillips, 1998) predict results consistent with experimental data, but neither is applicable in moderate to strong fluctuations regimes.

The Born approximation (de Wolf, 1965) is a perturbation technique and remains valid only as long as the amplitude fluctuations remain small. This approximation assumes that the field at the receiver can be calculated as a sum of the original incident field, $U_0 = A_0 \exp[j\phi_0]$, plus the field scattered one time from a turbulent blob, $U_1 = A_1 \exp[jS_1]$. It is assumed that the real and imaginary parts of U_1 are uncorrelated and have equal variances, so U_1 is said to be circular complex Gaussian. Thus, from the first-order Born approximation, the irradiance of the field along the optical axis, I, has, from (Andrews & Phillips, 1998), a pdf given by the modified Rice-Nakagami distribution,

$$f_I(I) = \frac{1}{2b_0} \exp\left[-\frac{(A_0^2 + I)}{2b_0}\right] I_0\left(\frac{2A_0}{2b_0}\sqrt{I}\right), \qquad I > 0, \tag{1}$$

where $2b_0{=}E[A_1^2]$ and the operator $E[\cdot]$ stands for ensemble average, being $I_0(\cdot)$ the modified Bessel function of the first kind and order zero. As shown above, the Born approximation includes only single scattering effects. However, for many problems in line-of-sight propagation, multiple scattering effects cannot be ignored and so, the results based on the Born approximation have a limited range of applicability, particularly at optical wavelengths. Due to the problems associated to the Born approximation, greater attention was focused on the Rytov method for optical wave propagation. Rytov's method is similar to the Born approximation in that it is a perturbation technique, but applied to a transformation of the scalar wave equation (Andrews & Phillips, 1998; de Wolf, 1965). It does satisfy one of the mentioned objections to the Born approximation in that it includes multiple scattering effects (Heidbreder, 1967). However, these effects are incorporated in an inflexible way which does not depend on the turbulence or other obvious factors. The method does contain both the Born approximation and geometrical optics as special cases, but does not extend the limitations on these methods as much as originally claimed. In this approach, the electric field is written as a product of the free-space field, U_0, and a complex-phase exponential, $\exp(\Psi)$. Based on the assumption that the first-order Born approximation, U_1, is a circular complex Gaussian random variable, it follows that so is the first-order Rytov approximation, $\Psi = \chi + jS$, where χ and S denote the first-order log-amplitude and phase, respectively, of the field. Then, the irradiance of the field at a given propagation distance can be expressed as:

$$I = |U_0|^2 \exp(\Psi + \Psi^*) = I_0 \exp(2\chi), \tag{2}$$

as was written in (Andrews & Phillips, 1998). In Eq. (2), $I_0 = |A_0|^2$ is the level of irradiance fluctuation in the absence of air turbulence that ensures that the fading does not attenuate or amplify the average power, i.e., $E[I] = |A_0|^2$. This may be thought of as a conservation of energy consideration and requires the choice of $E[\chi] = -\sigma_\chi^2$, as was explained in (Fried, 1967; Strohbehn, 1978), where $E[\chi]$ is the ensemble average of log-amplitude, whereas σ_χ^2 is its variance. By virtue of the central limit theorem, the marginal distribution of the log-amplitude is Gaussian distributed. Hence, from the Jacobian statistical transformation, the probability density function of the intensity can be identified to have a lognormal distribution

$$f_I(I) = \frac{1}{2I}\frac{1}{\sqrt{2\pi\sigma_\chi^2}}\exp\left(-\frac{\left[\ln\left(I/I_0\right)+2\sigma_\chi^2\right]^2}{8\sigma_\chi^2}\right), \qquad I>0, \tag{3}$$

as indicated in (Andrews & Phillips, 1998). Nevertheless, it has also been observed that the lognormal distribution can underestimate both the peak of the pdf and the behavior in the tails as compared with measured data (Churnside & Frehlich, 1989; Hill & Frehlich, 1997).
As the strength of turbulence increases and multiple self-interference effects must be taken into account, greater deviations from lognormal statistics are present in measured data. In fact, it has been predicted that the probability density function of irradiance should approach a negative exponential in the limit of infinite turbulence (Fante, 1975; de Wolf, 1974). The negative exponential distribution is considered a limit distribution for the irradiance and it is therefore approached only far into the saturation regime.

2.2 Modulated probability distribution functions

Early theoretical models developed for the irradiance fluctuations were based on assumptions of statistical homogeneity and isotropy. However, it is well known that atmospheric turbulence always contains large-scale components that usually destroy the homogeneity and isotropy of the meteorological fields, causing them to be non-stationary. This non-stationary nature of atmospheric turbulence has led to model optical scintillations as a conditional random process (Al-Habash et al., 2001; Churnside & Clifford, 1987; Churnside & Frehlich, 1989; Fante, 1975; Hill & Frehlich, 1997; Strohbehn, 1978; Wang & Strohbehn, 1974; de Wolf, 1974), in which the irradiance can be written as a product of one term that arises from large-scale turbulent eddy effects by a second term that represent the statistically independent small-scale eddy effects.
One of the first attempts to gain wide acceptance for a variety of applications was the K distribution (Abdi & Kaveh, 1998; Jakerman, 1980) that provides excellent models for predicting irradiance statistics in a variety of experiments involving radiation scattered by turbulent media. The K distribution can be derived from a mixture of the conditional negative exponential distribution and a gamma distribution. In particular, in this modulation process, the irradiance is assumed governed by the conditional negative exponential distribution:

$$f_1(I|b) = \frac{1}{b}\exp\left(-\frac{I}{b}\right), \qquad I>0, \tag{4}$$

as written in (Andrews & Phillips, 1998); whereas the mean irradiance, $b = E[I]$, is itself a random quantity assumed to be characterized by a gamma distribution given by

$$f_2(b) = \frac{\alpha(\alpha b)^{\alpha-1}}{\Gamma(\alpha)}\exp\left(-\alpha b\right), \qquad b>0, \quad \alpha>0. \tag{5}$$

In Eq. (5), $\Gamma(\cdot)$ is the gamma function and α is a positive parameter related to the effective number of discrete scatterers. The unconditional pdf for the irradiance is obtained by calculating the mixture of the two distributions presented above, and the resulting distribution is given by:

$$f_I(I) = \int_0^\infty f_1(I|b)f_2(b)\mathrm{d}b = \frac{2\alpha}{\Gamma(\alpha)}(\alpha I)^{\frac{(\alpha-1)}{2}}K_{\alpha-1}\left(2\sqrt{\alpha I}\right), \qquad I>0, \quad \alpha>0; \tag{6}$$

as detailed in (Andrews & Phillips, 1998). In Eq. (6), $K_p(x)$ is the modified Bessel function of the second kind and order p. The normalized variance of irradiance, commonly called the scintillation index, predicted by the K distribution satisfies $\sigma_I^2 = 1 + 2/\alpha$, which always exceeds unity but approaches it in the limit $\alpha \to \infty$. This fact restricts the usefulness of this distribution to moderate or strong turbulence regimes; even where it can be applied it tends to underestimate the probability of high irradiances (Churnside & Clifford, 1987) and, thus, to underestimate higher-order moments. Certainly, it is not valid under weak turbulence for which the scintillation index is less than unity. One attempt at extending the K distribution to the case of weak fluctuations led to the homodyned K (HK) (Jakerman, 1980) and the I-K distribution (Andrews & Phillips, 1985; 1986), this latter with a behavior very much like the HK distribution (Andrews & Phillips, 1986), but it did not generally provide a good fit to the experimental data in extended turbulence (Churnside & Frehlich, 1989).

With respect to other models based on modulation process, Wang and Strohbehn (Wang & Strohbehn, 1974) proposed a distribution, called log-normal Rician (LR) or also Beckmann's pdf, which results from the product of a Rician amplitude and a lognormal modulation factor. Thus, the observed field can be expressed, from (Churnside & Clifford, 1987), as:

$$U = (U_C + U_G)\exp(\chi + jS), \tag{7}$$

where U_C is a deterministic quantity and U_G is a circular Gaussian complex random variable, with χ and S being the log-amplitude and phase, respectively, of the field, assumed to be real Gaussian random variables. The irradiance is therefore given by $I = |U_C + U_G|^2 \exp(2\chi)$, where $|U_C + U_G|$ has a Rice-Nakagami pdf and the multiplicative perturbation, $\exp(2\chi)$, is lognormal. Then, the pdf is defined by the integral:

$$f_I(I) = \int_0^\infty f(I|\exp[2\chi])\, f(\exp[2\chi])\mathrm{d}[\exp(2\chi)], \tag{8}$$

where $f(I|\exp[2\chi])$ is the conditional probability density function of the irradiance given the perturbation $\exp(2\chi)$, governed by a Rician distribution; whereas $f(\exp[2\chi])$ denotes the lognormal pdf for the multiplicative perturbation. Then, Eq. (8) can be expressed as (Al-Habash et al., 2001):

$$f_I(I) = \frac{(1+r)\exp(-r)}{\sqrt{2\pi}\sigma_z} \int_0^\infty I_0\left\{2\left[\frac{(1+r)rI}{z}\right]^{1/2}\right\} \exp\left\{-\frac{(1+r)I}{z} - \frac{[\ln z + (1/2)\sigma_z^2]^2}{2\sigma_z^2}\right\}\frac{\mathrm{d}z}{z^2}, \tag{9}$$

where $r = |U_C|^2/|U_G|^2$ is the coherence parameter, z and σ_z^2 represent the irradiance modulation factor, $\exp(2\chi)$, and its variance, respectively, and $I_0(\cdot)$ is the zero-order modified Bessel function of the first kind. Although it provides an excellent fit to various experimental data, the LR pdf has certain impediments, for instance, a closed-form solution for this integral is unknown or its poor convergence properties that makes the LR model cumbersome for numerical calculations.

Under strong fluctuations, the LR model reduces to the lognormally modulated exponential distribution (Churnside & Hill, 1987), but this latter distribution is valid only under strong fluctuation conditions.

Finally, in a recent series of papers on scintillation theory (Al-Habash et al., 2001; Andrews et al., 1999), Andrews et al. introduced the modified Rytov theory and proposed the gamma-gamma pdf as a tractable mathematical model for atmospheric turbulence. This model is, again, a two-parameter distribution which is based on a doubly stochastic theory of scintillation and assumes that small scale irradiance fluctuations are modulated by large-scale irradiance fluctuations of the propagating wave, both governed by independent gamma distributions. Then, from the modified Rytov theory (Andrews et al., 1999), the optical field is defined as $U = U_0 \exp(\Psi_x + \Psi_y)$, where Ψ_x and Ψ_y are statistically independent complex perturbations which are due only to large-scale and small-scale atmospheric effects, respectively. Then, the irradiance is now defined as the product of two random processes, i.e., $I = I_x I_y$, where I_x and I_y arise, respectively, from large-scale and small-scale atmospheric effects. Moreover, both large-scale and small-scale irradiance fluctuations are governed by gamma distributions, i.e.:

$$f_x(I_x) = \frac{\alpha(\alpha I_x)^{\alpha-1}}{\Gamma(\alpha)} \exp\left(-\alpha I_x\right), \qquad I_x > 0, \quad \alpha > 0, \tag{10}$$

$$f_y(I_y) = \frac{\beta(\beta I_y)^{\beta-1}}{\Gamma(\beta)} \exp\left(-\beta I_y\right), \qquad I_y > 0, \quad \beta > 0. \tag{11}$$

To obtain the unconditional gamma-gamma irradiance distribution, we can form:

$$f_I(I) = \int_0^\infty f_y(I|I_x) f_x(I_x) \mathrm{d}I_x = \frac{2(\alpha\beta)^{(\alpha+\beta)/2}}{\Gamma(\alpha)\Gamma(\beta)} I^{(\alpha+\beta)/2-1} K_{\alpha-\beta}\left(2\sqrt{\alpha\beta I}\right), \qquad I > 0, \tag{12}$$

where $K_a(\cdot)$ is the modified Bessel function of the second kind of order a. In Eq. (12), the positive parameter α represents the effective number of large-scale cells of the scattering process, larger than that of the first Fresnel zone or the scattering disk whichever is larger (Al-Habash et al., 2001); whereas β similarly represents the effective number of small-scale cells, smaller than the Fresnel zone or the coherence radius. This gamma-gamma pdf has been suggested as a reasonable alternative to Beckmann's pdf because makes computations easier in comparison with this latter distribution.

Now, through this chapter, we propose a new and generic propagation model and, from it, and assuming a gamma approximation for the large-scale fluctuations, we obtain a new and unifying statistical model for the irradiance fluctuations. The proposed model is valid under all range of turbulence conditions (weak to strong) and it is found to provide an excellent fit to the experimental data, as will be shown through Section 5. Furthermore, the statistical model presented in this chapter can be written in a closed-form expression and it contains most of the statistical models for the irradiance fluctuations that have been proposed in the bibliography.

3. Generation of a new distribution: the $\mathcal{M}$ distribution

As was pointed out before, the Rytov theory is the conventional method of analysis in weak-fluctuations regimes, as shown in Eq. (2). Extensions to such theory were developed in (Churnside & Clifford, 1987; Wang & Strohbehn, 1974) to obtain the LR model; and in (Al-Habash et al., 2001) to generate the gamma-gamma pdf as a plausible and easily tractable approximation to Beckmann's pdf. Both models, of course, approximate the behavior of optical irradiance fluctuations in the turbulent atmosphere under all irradiance fluctuation regimes. In fact, the LR model can be seen as a generic model because includes the lognormal distribution which can be employed under weak turbulence; the lognormally modulated exponential distribution used in strong path-integrated turbulence and, moreover, it can be reduced to the negative exponential pdf in extremely strong turbulence regimes (Churnside & Frehlich, 1989). On this basis, we propose a more generic distribution model that includes, as special cases, almost all valid models and theories that have been previously proposed in the bibliography, unifying them in a more general closed-form formulation. Thus, among others, the Rice-Nakagami, the lognormal, the K and the HK distribution, the gamma-gamma and the negative-exponential models are contained and, as we detail through this chapter, a gamma-Rician distribution can be derived from our proposed model as a very accurate alternative to the LR pdf for its simple closed-form representation (we must remark that a closed-form solution for the LR pdf is still unknown).

3.1 The model of propagation including a new scattering component coupled to the line-of-sight contribution

Assume an electromagnetic wave is propagating through a turbulent atmosphere with a random refractive index. As the wave passes through this medium, part of the energy is scattered and the form of the irradiance probability distribution is determined by the type of scattering involved. In the physical model we present in this chapter, the observed field at the receiver consists of three terms: the first one is the line-of-sight (LOS) contribution, U_L, the second one is the component which is quasi-forward scattered by the eddies on the propagation axis, U_S^C and coupled to the LOS contribution; whereas the third term, U_S^G, is due to energy which is scattered to the receiver by off-axis eddies, this latter contribution being statistically independent from the previous two other terms. The inclusion of this coupled to the LOS scattering component is the main novelty of the model and it can be justified by the high directivity and the narrow beamwidths of laser beams in atmospheric optical communications. The model description is depicted in Fig. 1.

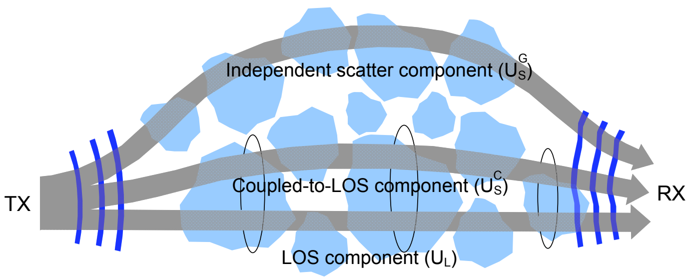


Fig. 1. Proposed propagation geometry for a laser beam where the observed field at the receiver consists of three terms: first, the line-of-sight (LOS) component, U_L; the second term is the coupled-to-LOS scattering term, U_S^C, whereas the third path represents the energy scattered to the receiver by off-axis eddies, U_S^G.

Mathematically, we can write the total observed field as:

$$U = \left(U_L + U_S^C + U_S^G\right) \exp\left(\chi + jS\right) \tag{13}$$

where

$$U_L = \sqrt{G}\sqrt{\Omega} \exp\left(j\phi_A\right), \tag{14}$$

$$U_S^C = \sqrt{\rho}\sqrt{G}\sqrt{2b_0} \exp\left(j\phi_B\right), \tag{15}$$

$$U_S^G = \sqrt{(1-\rho)U_S'}; \tag{16}$$

being U_S^C and U_S^G statistically independent stationary random processes. Of course, U_L and U_S^G are also independent random processes. In Eq. (13), G is a real variable following a gamma distribution with $E[G] = 1$. It represents the slow fluctuation of the LOS component. Following the same notation as (Abdi et al., 2003), the parameter $\Omega = E[|U_L|^2]$ represents the average power of the LOS component whereas the average power of the total scatter components is denoted by $2b_0 = E[|U_S^C|^2 + |U_S^G|^2]$. ϕ_A and ϕ_B are the deterministic phases of the LOS and the coupled-to-LOS scatter components, respectively. On another note, $0 \leq \rho \leq 1$ is the factor expressing the amount of scattering power coupled to the LOS component. This ρ factor depends on the propagation path length, L, the intensity of the turbulence, the optical wavelength, λ, the beam diameter, the average scale of inhomogeneities ($l = \sqrt{\lambda L}$), the beam divergence due to the atmospheric-induced beam spreading, and the distance between the different propagation paths (line of sight component and scattering components), due to if the spacing between such paths is greater than the fading correlation length, then turbulence-induced fading is uncorrelated. Finally, U_S' is a circular Gaussian complex random variable, and χ and S are, again, real random variables representing the log-amplitude and phase perturbation of the field induced by the atmospheric turbulence, respectively.

As an advance, the proposed model, with the inclusion of a random nature in the LOS component in addition to a new scattering contribution coupled to the LOS component, offers a highly positive mathematical conditioning due to its obtained irradiance pdf can be expressed in a closed-form expression and it approaches as much as desired to the result derived from the LR model, for which a closed-form solution for its integral is still unknown. Moreover, it has a high level of generality due to it includes as special cases most of the distribution models proposed in the bibliography until now.

From Eq. (13), the irradiance is therefore given by:

$$\begin{aligned} I =& \left| U_L + U_S^C + U_S^G \right|^2 \exp(2\chi) = \\ =& \left| \sqrt{G}\sqrt{\Omega}\exp(j\phi_A) + \sqrt{\rho}\sqrt{G}\sqrt{2b_0}\exp(j\phi_B) + \sqrt{(1-\rho)}U_S' \right|^2 \exp(2\chi). \end{aligned} \tag{17}$$

As indicated in (Churnside & Clifford, 1987), the larger eddies in the atmosphere produce the lognormal statistics and the smaller ones produce the shadowed-Rice model analogous to the one proposed in (Abdi et al., 2003).

As was explained in (Wang & Strohbehn, 1974), there is no strong physical justification for choosing a particular propagation model and different forms could be chosen equally well. However, there exists some points to support our proposal: so if we assume the conservation of energy consideration, then $E[I] = \Omega + 2b_0$ and requires the choice of $E[\chi] = -\sigma_\chi^2$, as was detailed in (Fried, 1967; Strohbehn, 1978). Finally, a plausible justification for the coupled-to-LOS scattering component, U_S^C, is provided in (Kennedy, 1970). There, it is said that if the turbulent medium is so thin that multiple scattering can be ignored, the multipath delays of the scattered radiation collected by a diffraction-limited receiver will usually be small relative to the signal bandwidth. Then the scattered field will combine coherently with the unscattered field and there will be no-"interfering" signal component of the field, in a similar way as U_S^C combines with U_L in our proposed model. Of course, when the turbulent medium becomes so thick, then the unscattered component of the field can be neglected.

3.2 Málaga ($\mathcal{M}$) probability density function

From Eq. (17), the observed irradiance of our proposed propagation model can be written as:

$$I = \left|U_L + U_S^C + U_S^G\right|^2 \exp(2\chi) = YX,$$
$$\begin{cases} Y \triangleq \left|U_L + U_S^C + U_S^G\right|^2 & \text{(small-scale fluctuations)} \\ X \triangleq \exp(2\chi) & \text{(large-scale fluctuations)}, \end{cases} \tag{18}$$

where the small-scale fluctuations denotes the small-scale contributions to scintillation associated with turbulent cells smaller than either the first Fresnel zone or the transverse spatial coherence radius, whichever is smallest. In contrast, large-scale fluctuations of the irradiance are generated by turbulent cells larger than that of either the Fresnel zone or the so-called "scattering disk", whichever is largest. From Eq. (13), we rewrite the lowpass-equivalent complex envelope as:

$$R(t) = \left(U_L + U_S^C + U_S^G\right) = \sqrt{G}\left(\sqrt{\Omega}\exp(j\phi_A) + \sqrt{\rho}\sqrt{2b_0}\exp(j\phi_B)\right) + \sqrt{(1-\rho)}U_S', \tag{19}$$

so that we have the identical shadowed Rice single model employed in (Abdi et al., 2003), composed by the sum of a Rayleigh random phasor (the independent scatter component, U_S') and a Nakagami distribution ($\sqrt{G}$, used for both the LOS component and the coupled-to-LOS scatter component). The other remaining terms in Eq. (19) are deterministics. Then, we can apply the same procedure exposed in (Abdi et al., 2003) consisting in calculating the expectation of the Rayleigh component with respect to the Nakagami distribution and then deriving the pdf of the instantaneous power. Hence, the pdf of Y is given by:

$$f_Y(y) = \frac{1}{\gamma}\left[\frac{\gamma\beta}{\gamma\beta + \Omega'}\right]^{\beta} \exp\left[-\frac{y}{\gamma}\right] {}_1F_1\left(\beta; 1; \frac{1}{\gamma}\frac{\Omega'}{(\gamma\beta + \Omega')}y\right), \tag{20}$$

where $\beta \triangleq (E[G])^2/\mathrm{Var}[G]$ is the amount of fading parameter with $\mathrm{Var}[\cdot]$ as the variance operator. We have denoted $\Omega' = \Omega + \rho 2b_0 + 2\sqrt{2b_0\Omega\rho}\cos(\phi_A - \phi_B)$ and $\gamma = 2b_0(1-\rho)$. Finally, ${}_1F_1(a; c; x)$ is the Kummer confluent hypergeometric function of the first kind.

Otherwise, the large-scale fluctuations, $X \triangleq \exp(2\chi)$, is widely accepted to be a lognormal amplitude (Churnside & Clifford, 1987) but, however, as in (Abdi et al., 2003; Al-Habash et al., 2001; Andrews & Phillips, 2008; Phillips & Andrews, 1982), this distribution is approximated by a gamma one, this latter with a more favorable analytical structure. This latter distribution can exhibit characteristics of the lognormal distribution under the proper conditions, avoiding the infinite-range integral of the lognormal pdf. Then, the gamma pdf is given by:

$$f_X(x) = \frac{\alpha^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\alpha x), \tag{21}$$

where α is a positive parameter related to the effective number of large-scale cells of the scattering process, as in (Al-Habash et al., 2001). Now, the statistical characterization of the model presented in Eq. (17) will be formally accomplished.S

Definition: *Let $I{=}XY$ be a random variable representing the irradiance fluctuations for a propagating optical wave. It is said that I follows a generalized $\mathcal{M}$ distribution if X and Y are random variable distributions according to Eqs.* (21) *and* (20)*, respectively. That the distribution of I is a generalized $\mathcal{M}$ distribution can be written in the following notation: $I \sim \mathcal{M}^{(G)}(\alpha, \beta, \gamma, \rho, \Omega')$, being $\alpha, \beta, \gamma, \rho, \Omega'$ the real and positive parameters of this generalized $\mathcal{M}$ distribution. And for the pivotal case of β being a natural number, then it is said that I follows an $\mathcal{M}$ distribution and it is denoted by $\mathcal{M}(\alpha, \beta, \gamma, \rho, \Omega')$.*

Lemma 1: *Let $I \sim \mathcal{M}^{(G)}(\alpha, \beta, \gamma, \rho, \Omega^{'})$. Then, its pdf is represented by:*

$$f_I(I) = A^{(G)} \sum_{k=1}^{\infty} a_k^{(G)} I^{\frac{\alpha+k}{2}-1} K_{\alpha-k}\left(2\sqrt{\frac{\alpha I}{\gamma}}\right), \tag{22}$$

where

$$\begin{cases} A^{(G)} \triangleq \dfrac{2\alpha^{\frac{\alpha}{2}}}{\gamma^{1+\frac{\alpha}{2}}\Gamma(\alpha)}\left(\dfrac{\gamma\beta}{\gamma\beta+\Omega^{'}}\right)^{\beta}; \\ a_k^{(G)} \triangleq \dfrac{(\beta)_{k-1}(\alpha\gamma)^{\frac{k}{2}}}{\left[(k-1)!\right]^2 \gamma^{k-1}\left(\Omega^{'}+\gamma\beta\right)^{k-1}}. \end{cases} \tag{23}$$

In Eq. (22), *$K_\nu(\cdot)$ is the modified Bessel function of the second kind and order ν whereas $\Gamma(\cdot)$ is the gamma function.*
Otherwise, let $I \sim \mathcal{M}(\alpha, \beta, \gamma, \rho, \Omega^{'})$, i.e., β is a natural number; then, its pdf is given by:

$$f_I(I) = A \sum_{k=1}^{\beta} a_k I^{\frac{\alpha+k}{2}-1} K_{\alpha-k}\left(2\sqrt{\frac{\alpha\beta I}{\gamma\beta+\Omega^{'}}}\right) \tag{24}$$

where

$$\begin{cases} A \triangleq A^{(G)}\left(\dfrac{\gamma\beta}{\gamma\beta+\Omega^{'}}\right)^{\frac{\alpha}{2}}; \\ a_k \triangleq \dbinom{\beta-1}{k-1}\dfrac{1}{(k-1)!}\left(\dfrac{\Omega^{'}}{\gamma}\right)^{k-1}\left(\dfrac{\alpha}{\beta}\right)^{\frac{k}{2}}\left(\gamma\beta+\Omega^{'}\right)^{1-\frac{k}{2}}. \end{cases} \tag{25}$$

In Eq. (24), *$K_\nu(\cdot)$ is, again, the modified Bessel function of the second kind and order ν. Moreover, in Eq.* (25), *$A^{(G)}$ was one of the parameters defined in Eq.* (23), *whereas $\binom{\beta}{k}$ represents the binomial coefficient.* In the interest of clarity, the proof of this lemma is moved to Appendix A.
To conclude this subsection, we can point out that the pdf functions given in Eq. (22) and Eq. (24) can be expressed as a discrete mixture and a finite discrete mixture, respectively (see Chap. 7 of Ref. (Charalambides, 2005)) involving a resized irradiance variable, I', in the form: $f_{I'}(I') = \sum_k \omega_k \cdot f_{GG}(I')$, being the mixed distribution, $f_{GG}(I')$, a gamma-gamma pdf whereas the weight function, ω_k, satisfies that $\sum_k \omega_k = 1$ due to $\int_0^\infty f_{I'}(y)\mathrm{d}y = \sum_k \omega_k \cdot \int_0^\infty f_{GG}(y)\mathrm{d}y = 1$ by definition and $\int_0^\infty f_{GG}(y)\mathrm{d}y$=1 also by definition.

3.3 Moments of the $\mathcal{M}$ probability distribution

In this subsection, the k^{th} moment of the $\mathcal{M}$ probability distribution is obtained.
Lemma 2: *Let I the randomly fading irradiance signal following a generalized $\mathcal{M}$ distribution and expressed as $I \sim \mathcal{M}^{(G)}(\alpha, \beta, \gamma, \rho, \Omega^{'})$. Then, its centered moments, denoted by $m_k^{(G)}(I)$, are given by:*

$$m_k^{(G)}(I) \triangleq E\left[I^k\right] = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\alpha^k}\left(\frac{\gamma\beta}{\gamma\beta+\Omega^{'}}\right)^{\beta}\gamma^k\Gamma(k+1)\,{}_2F_1\left(k+1,\beta;1;\frac{\Omega^{'}}{\gamma\beta+\Omega^{'}}\right), \tag{26}$$

where ${}_2F_1(a,b;c;x)$ is the Gaussian hypergeometric function. In addition, if the intensity signal now follows an $\mathcal{M}$ distribution, $I \sim \mathcal{M}(\alpha, \beta, \gamma, \rho, \Omega^{'})$, with β being a natural number, then its centered *moments are given by:*

$$m_k\left(I\right)=\frac{\Gamma\left(\alpha+k\right)}{\Gamma\left(\alpha\right)\alpha^k}\frac{1}{\gamma}\left(\frac{\gamma\beta}{\gamma\beta+\Omega^{'}}\right)^{\beta}\sum_{r=0}^{\beta-1}\binom{\beta-1}{r}\frac{1}{r!}\left(\frac{\Omega^{'}}{\gamma\left(\gamma\beta+\Omega^{'}\right)}\right)^{r}\frac{\Gamma\left(k+r+1\right)}{\left(\frac{\beta}{\gamma\beta+\Omega^{'}}\right)^{k+r+1}}. \tag{27}$$

For the sake of clarity of the whole chapter, the proof of this lemma is, again, moved and extensively explained in Appendix B.

3.4 Cumulative distribution function (cdf) of the $\mathcal{M}$ probability distribution

In this subsection, the cumulative distribution function (cdf) of the $\mathcal{M}$ probability distribution is obtained.

Lemma 3: *Let I the randomly fading irradiance signal following a generalized $\mathcal{M}$ distribution and expressed as $I\sim\mathcal{M}^{(G)}(\alpha,\beta,\gamma,\rho,\Omega^{'})$. Then, its cdf is given by:*

$$P(I\le I_T)=\int_0^{I_T}f_I(I)\mathrm{d}I=\frac{A^{(G)}}{I_T^{\frac{\alpha}{2}+1}}\times$$
$$\sum_{k=1}^{\infty}\frac{a_k^{(G)}}{I_T^{\frac{k}{2}}}\left\{2^{-(\alpha-k)-1}\left(2I_T^{-1/2}\sqrt{\frac{\alpha}{\gamma}}\right)^{-(\alpha-k)}\frac{\Gamma(\alpha-k)}{k+1}\,{}_1F_2\left(k+1;1-\alpha+k,k+2;\frac{\alpha}{\gamma I_T}\right)+\right.$$
$$\left.+2^{1-(\alpha-k)}\left(2I_T^{-1/2}\sqrt{\frac{\alpha}{\gamma}}\right)^{(\alpha-k)}\frac{\Gamma(k-\alpha)}{\alpha+1}\,{}_1F_2\left(\alpha+1;1+\alpha-k,\alpha+2;\frac{\alpha}{\gamma I_T}\right)\right\}, \tag{28}$$

where I_T is a threshold parameter, $A^{(G)}$ and $a_k^{(G)}$ are defined in Eq. (23) *and ${}_1F_2(a;c,d;x)$ deno-tes a generalized hypergeometric function. Nevertheless, if the irradiance signal now follows an $\mathcal{M}$ distribution, $I\sim\mathcal{M}(\alpha,\beta,\gamma,\rho,\Omega^{'})$, with β being a natural number, then its cdf is given by:*

$$P(I\le I_T)=\int_0^{I_T}f_I(I)\mathrm{d}I=\frac{A}{I_T^{\frac{\alpha}{2}+1}}\times$$
$$\sum_{k=1}^{\beta}\frac{a_k}{I_T^{\frac{k}{2}}}\left\{2^{-(\alpha-k)-1}\left(\frac{2I_T^{-\frac{1}{2}}}{k+1}\sqrt{\frac{\alpha\beta}{\gamma\beta+\Omega^{'}}}\right)^{-(\alpha-k)}\Gamma(\alpha-k)\,{}_1F_2\left(k+1;1-\alpha+k,k+2;\frac{\alpha\beta}{\left(\gamma\beta+\Omega^{'}\right)I_T}\right)+\right.$$
$$\left.+2^{1-(\alpha-k)}\left(\frac{2I_T^{-\frac{1}{2}}}{\alpha+1}\sqrt{\frac{\alpha\beta}{\gamma\beta+\Omega^{'}}}\right)^{(\alpha-k)}\Gamma(k-\alpha)\,{}_1F_2\left(\alpha+1;1+\alpha-k,\alpha+2;\frac{\alpha\beta}{\left(\gamma\beta+\Omega^{'}\right)I_T}\right)\right\}, \tag{29}$$

where, again, I_T is a threshold parameter and A and a_k are defined in Eq. (25). The proof of this lemma is treated in Appendix C.

4. Derivation of existing distribution models

In this section, we derive, from our proposed generalized distribution, $\mathcal{M}^{(G)}(\alpha,\beta,\gamma,\rho,\Omega^{'})$, (or from $\mathcal{M}(\alpha,\beta,\gamma,\rho,\Omega^{'})$, if its β parameter is a natural number) most of the existing distribution models that have been proposed for atmospheric optical communications in the bibliography.

4.1 Rice-Nakagami and lognormal distribution functions

Consider the propagation model presented in this chapter and written in Eq. (17). Thus, starting with the first models proposed in the bibliography for weak turbulence regimes, we indicated in Section 2 that, from the first-order Born approximation, the irradiance, I, has a pdf governed by the modified Rice-Nakagami distribution (see Eq. (1)). From Eq. (17), if we assume both $\rho = 0$ and Var$[|U_L|]$= 0, where Var$[\cdot]$ represents the variance operator, then U_L becomes a constant random variable where $E[|U_L|] = \sqrt{\Omega}$ since $E[G] = 1$, as was pointed out in Section 2. If we consider that χ is a zero mean random variable (strictly speaking, $E[\chi] = -\sigma_\chi^2$ due to conservation energy consideration (Fried, 1967; Strohbehn, 1978)) and Var$[\chi]$=Var$[S] = 0$, then, from (Andrews & Phillips, 1998), Eq.(17) becomes:

$$I = \left|\sqrt{G}\sqrt{\Omega}\exp\left(j\phi_A\right) + U_S'\right|^2. \tag{30}$$

Equation (30) represents the first-order Born approximation, as indicated in (Andrews & Phillips, 1998). As $U_S' = A_S' \exp\left(jS_S'\right)$ is a circular Gaussian complex random variable where $E[|U_S'|^2] = 2b_0 = \gamma$ owing to $\rho = 0$; and if we denote $A_0 = \sqrt{G}\sqrt{\Omega}$, then the irradiance, I, of the field along the optical axis has a modified Rice-Nakagami distribution given by:

$$f_I(I) = \frac{1}{\gamma}\exp\left[-\frac{\left(A_0^2 + I\right)}{\gamma}\right] I_0\left(\frac{2A_0}{\gamma}\sqrt{I}\right), \qquad I > 0, \tag{31}$$

identical to Eq. (1). Thus, the Rice-Nakagami distribution is included in our proposed $\mathcal{M}$ distribution. Moreover, as indicated in (Strohbehn, 1978), when $A_0^2/\gamma \to \infty$, then the Rice-Nakagami distribution leads to a lognormal distribution, one of the most widely employed distributions for weak turbulence regimes and derived by the used of the Rytov method and the application of the central limit theorem.

4.2 Rytov model

Thus, consider now the following different perturbational approach, the Rytov approximation, again restricted to weak fluctuation conditions. In this case, as was commented above, the pdf for the irradiance fluctuations is the lognormal distribution shown in Eq. (3). We can deduce this model from our proposed perturbation model written in Eqs. (13) and (17). Thus, lets assume again Var$[|U_L|]$= 0, so U_L becomes a constant random variable where $E[|U_L|]$=$\sqrt{\Omega}$ since $E[G]$=1 as was discussed in Section 2. If the average power of the total scatter components is established to $2b_0 = 0$ (no scattering power, $U_S^C = U_S^G = 0$), then Eq. (17) reduces to:

$$I = |U_L|^2 \exp\left(2\chi\right) = \left|\sqrt{G}\sqrt{\Omega}\exp\left(j\phi_A\right)\right|^2 \exp\left(2\chi\right). \tag{32}$$

If we identify I_0=$\left|\sqrt{G}\sqrt{\Omega}\exp\left(j\phi_A\right)\right|^2$ as the irradiance fluctuation in the absence of air turbulence and we assume the conservation of energy consideration $E[\chi]$=$-\sigma_\chi^2$, then we have the same conditions exposed in Eq. (2) so that the pdf of the intensity could be identified to have a lognormal distribution, as in Eq. (3). However, we have approximated the behavior of the large-scale fluctuations, $X = \exp\left(2\chi\right)$, by a gamma distribution due to it is proven that lognormal and gamma distributions can closely approximate each other (Clark & Karp, 1970). Thus, the behavior of the classical first-order Rytov approximation is included in our proposed propagation model.

4.3 Generation of existing modulated probability density functions

4.3.1 K, HK and negative exponential distribution

Now, to obtain the modulated probability distribution functions that have been widely employed in the bibliography, we must start calculating the moment generating function (MGF) of the random processes X and Y defined in Eq. (18). The MGF for a generic function, $f_Z(z)$, is defined by $M_Z(s) \triangleq \mathcal{L}\{f_Z(z); -s\}$, where $\mathcal{L}[\cdot]$ denotes the Laplace transform. Hence, from Eqs. (2.68) and (2.22) of Ref. (Simon & Alouini, 2005), we have:

$$M_Y(s) \triangleq \mathcal{L}[f_Y(y); -s] = \left(\frac{\gamma\beta}{\gamma\beta + \Omega'}\right)^{\beta} \frac{(1-\gamma s)^{\beta-1}}{\left(1 - \frac{\Omega'}{\gamma\beta+\Omega'} - \gamma s\right)^{\beta}}, \tag{33}$$

$$M_X(s) \triangleq \mathcal{L}[f_X(x); -s] = \frac{1}{(1-\frac{s}{\alpha})^{\alpha}}; \tag{34}$$

for $f_Y(y)$ and $f_X(x)$ given in Eqs. (20) and (21), respectively. Now, if $\Omega = 0$ (no LOS power) and $\rho = 0$ (no coupled-to-LOS scattering power, U_S^C), i.e., $\Omega^{'} = 0$, then Eq. (33) is reduced to:

$$M_Y(s) = (1-\gamma s)^{-1}, \tag{35}$$

and Eq. (20) is, obviously, reduced to an exponential distribution:

$$f_Y(y) = \frac{1}{\gamma} \exp\left[-\frac{y}{\gamma}\right]. \tag{36}$$

In addition, we can obtain this exponential distribution when β is unity in Eq. (33), and Eq. (36) would be written in the same form, replacing γ parameter by $\gamma + \Omega^{'}$. Anyhow, as was detailed in (Andrews & Phillips, 1998), with a negative exponential distribution for $f_Y(y)$ and a gamma distribution for $f_X(x)$, the unconditional pdf for the irradiance is obtained by calculating the mixture of these two latter distributions in the same form indicated in Eq. (6), leading to the K-distribution model. Of course, as the effective number of discrete scatterer cells, α, becomes unbounded (a huge thick turbulent medium), i.e., $\alpha \to \infty$, the K distribution tends to the negative exponential distribution as the gamma distribution that governs X approaches a delta function (Andrews et al., 2001). So the K distribution and the exponential one are also included in our proposed statistical model. Finally, a generalization of the K distribution, the homodyned K (HK) distribution is also included (Andrews & Phillips, 1986). This HK model is composed by a Rice-Nakagami distribution and a gamma distribution. The Rice-Nakagami model can be deduced in a similar way as Eq. (31). However, the gamma model needed to build the unconditional HK pdf is the distribution function of the fluctuating average irradiance of the random field component (U_S^G since U_S^C=0 as ρ=0 for deriving the Rice-Nakagami model from our $\mathcal{M}$ distribution). Thus, we have to identify the large-scale fluctuations, X, given in Eq. (18) with the parameter γ=$2b_0$=$E[|U_S^G|^2]$ so that $x \triangleq \gamma$ in Eq. (21). Then, the HK distribution is also contained in our proposed model as a special case of the $\mathcal{M}$ distribution.

4.3.2 Gamma-gamma model

On the other hand, and returning again to our original model given in Eqs. (20) and (21) with their MGFs calculated in Eqs. (33) and (34), we now take $\rho = 1$, i.e., there only exists LOS component, U_L, and coupled-to-LOS scattering component, U_S^C, in our propagation model given in Eq. (17). If $\rho = 1$ then $\gamma = 0$ so Eq. (33) becomes:

$$M_Y(s) = \lim_{\gamma \to 0} \left\{ \left(\frac{\gamma\beta}{\gamma\beta + \Omega'} \right)^{\beta} \frac{(1-\gamma s)^{\beta-1}}{\left(1 - \frac{\Omega'}{\gamma\beta+\Omega'} - \gamma s\right)^{\beta}} \right\} = \left(\Omega'\right)^{-\beta} \left(\frac{1}{\Omega'} - \frac{s}{\beta} \right)^{-\beta}. \tag{37}$$

If we fix $\Omega' = 1$, then Eq. (37) is reduced to:

$$M_Y(s) = \left(1 - \frac{s}{\beta}\right)^{-\beta}. \tag{38}$$

This last expression is the MGF of a gamma function so that we can identify that the small-scale fluctuations, Y, are governed by a gamma distribution. As the behavior of large-scale fluctuations, X were approximated to follow a gamma distribution, then the unconditional pdf for the irradiance is obtained by calculating the mixture of these two gamma distributions in the same form as indicated in Eq. (12). Then, the gamma-gamma model presented in (Al-Habash et al., 2001) is also included in our $\mathcal{M}$ model by, first, canceling the U_S^G component, i.e., the energy which is scattered to the receiver by off-axis eddies; and, secondly, normalizing the Ω component at 1. In this particular case, α represents the effective number of large-scale cells of the scattering process and β similarly represents the effective number of small-scale effects, in the same form as was explained in (Al-Habash et al., 2001).

4.3.3 Gamma-Rician model approximating to lognormal-Rician (LR) model

Finally, and again returning to our original model given in Eqs. (20) and (21) and in Eqs. (33) and (34), we can approximate our $\mathcal{M}$ distribution to the LR model proposed in Eq. (9). For this purpose, we only need to take $\beta \to \infty$; then, from the definition of $e = (1 + 1/x)^x$, $x \to \infty$, and from L'Hopital's rule, the MGF of Y is given by:

$$M_Y(s) = \lim_{\beta \to \infty} \left\{ \left(\frac{\gamma\beta}{\gamma\beta + \Omega'} \right)^{\beta} \frac{(1-\gamma s)^{\beta-1}}{\left(1 - \frac{\Omega'}{\gamma\beta+\Omega'} - \gamma s\right)^{\beta}} \right\} = (1-\gamma s)^{-1} \exp\left[\frac{\Omega' s}{1-\gamma s} \right], \tag{39}$$

according to Eq. (2.17) of Ref. (Simon & Alouini, 2005), where its associated pdf is, from Eq. (20), rewritten as:

$$f_Y(y) = \frac{1}{\gamma} \exp\left[-\frac{y + \Omega'}{\gamma} \right] I_0 \left(\frac{2\sqrt{\Omega' y}}{\gamma} \right). \tag{40}$$

Equation (40) represents a Rice pdf (Abdi et al., 2003). As the behavior of large-scale fluctuations, X were approximated to follow a gamma distribution as indicated in Eq. (21), then the unconditional pdf for the irradiance, I, is obtained by calculating the mixture of these two gamma distributions in the form:

$$f_I(I) = \int_0^\infty f_Y\left(I|x\right) f_X(x)\mathrm{d}x = \\ = \frac{1}{\gamma}\frac{\alpha^\alpha}{\Gamma(\alpha)}\exp\left(-\frac{\Omega^{'}}{\gamma}\right)\sum_{k=0}^{\infty}\frac{(-1)^k\left(\Omega^{'}I\right)^k}{k!\Gamma(k+1)\gamma^{2k}}\int_0^\infty x^{\alpha-2-k}\exp\left(-\frac{I}{x\gamma}-\alpha x\right)\mathrm{d}x, \tag{41}$$

where we have expanded the modified Bessel function, $I_0(\cdot)$, by its series representation:

$$I_p(z) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(z/2\right)^{2k+p}}{k!\Gamma(k+p+1)}, \qquad |z| < \infty; \tag{42}$$

as indicated in (Andrews, 1998). Now, using again Eq. (3.471-9) of Ref. (Gradshteyn & Ryzhik, 2000), written in this chapter in Eq. (52), and substituting it into Eq. (41), we can derive:

$$f_I(I) = \frac{1}{\gamma}\frac{\alpha^\alpha}{\Gamma(\alpha)}\exp\left(-\frac{\Omega^{'}}{\gamma}\right)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\left(\Omega^{'}I\right)^{k-1}}{(k-1)!\Gamma(k)\gamma^{2k-2}}\left(\frac{I}{\alpha\gamma}\right)^{\frac{\alpha-k}{2}}K_{\alpha-k}\left(2\sqrt{\frac{\alpha I}{\gamma}}\right). \tag{43}$$

On the other hand, Eq. (43) can be expressed as:

$$f_I(I) = \hat{A}\sum_{k=1}^{\infty}\hat{a_k}I^{\frac{\alpha+k}{2}-1}K_{\alpha-k}\left(2\sqrt{\frac{\alpha I}{\gamma}}\right), \tag{44}$$

where

$$\begin{cases} \hat{A} \triangleq \dfrac{2\alpha^{\frac{\alpha}{2}}}{\gamma^{1+\frac{\alpha}{2}}\Gamma\left(\alpha\right)}\exp\left(-\dfrac{\Omega^{'}}{\gamma}\right); \\ \hat{a_k} \triangleq \dfrac{(-1)^{k-1}{\Omega^{'}}^{k-1}\left(\alpha\gamma\right)^{\frac{k}{2}}}{(k-1)!\Gamma(k)\gamma^{2k-2}}. \end{cases} \tag{45}$$

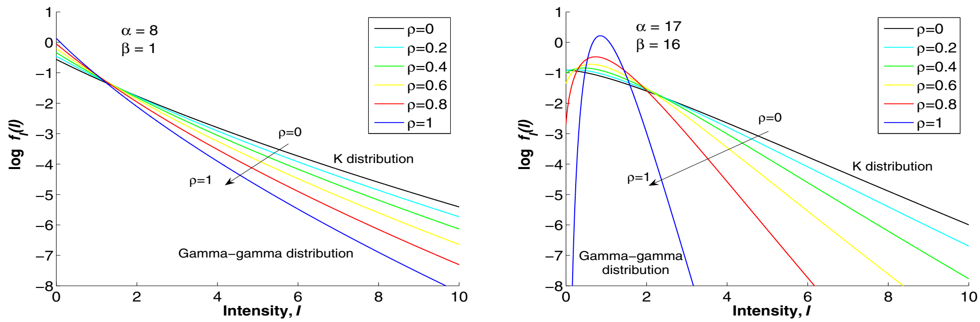
Then, the distribution directly derived from our proposed $\mathcal{M}$-distribution when $\beta \to \infty$ and presented in Eq. (43) is a gamma-Rician model. Of course, this gamma-Rician distribution is suggested to approximate the LR model detailed in (Churnside & Clifford, 1987), in which the large-scale fluctuations, X, are assumed to follow a lognormal distribution. But, as was discussed in Section 3.2., a lognormal distribution is well approximated by a gamma one (Abdi et al., 2003; Al-Habash et al., 2001; Andrews & Phillips, 2008). So this gamma-Rician approximation to the LR model will provide an excellent fit to experimental data avoiding the impediments of the LR model; thus, the gamma-Rician approximation provides a closed-form solution whereas the solution to the integral in the LR model is unknown and, moreover, its integral form undergoes a poor convergence making the LR pdf cumbersome for numerical calculations. In addition, the gamma-Rician approximation derived from our proposed $\mathcal{M}$ distribution has directly identified the α parameter, related to the large-scale cells of the scattering process, as in the gamma-gamma distribution (Abdi et al., 2003); whereas the other parameters can be calculated by using the heuristic theory of Clifford *et al.*, (Clifford et al., 1974), Hill and Clifford, (Hill & Clifford, 1981) and Hill (Hill, 1982).

Distribution model	Generation	Distribution model	Generation
Rice-Nakagami	$\rho = 0$ $\text{Var}[\lvert U_L \rvert] = 0$	Lognormal	$\rho = 0$ $\text{Var}[\lvert U_L \rvert] = 0$ $\gamma \to 0$
Gamma	$\rho = 0$ $\gamma = 0$	K distribution	$\Omega = 0$ and $\rho = 0$ or $\beta = 1$
HK distribution	$\text{Var}[G] = 0$ $\rho = 0$ $X = \gamma$	Exponential distribution	$\Omega = 0$ $\rho = 0$ $\alpha \to \infty$
Gamma-gamma distribution	$\rho = 1$, then $\gamma = 0$ $\Omega' = 1$	Gamma-Rician distribution	$\beta \to \infty$
Shadowed-Rician distribution	$\text{Var}[\lvert X \rvert] = 0$		

Table 1. List of existing distribution models for atmospheric optical communications and generation by using the proposed $\mathcal{M}$ distribution model.

4.4 Summary

To conclude this section, all the approximations involved in deriving the different distribution models that, until now, had been proposed in the bibliography are summarize in Table 1. Finally, Fig. 2 displays, as an example, the K distribution and the gamma-gamma one as special cases of the $\mathcal{M}$ distribution, showing the transition between them corresponding to various values of the factor ρ representing the amount of scattering power coupled to the LOS component. In such example, we have fixed $\Omega = 0$, $2b_0 = 1$ and $\phi_A - \phi_B = \pi/2$.



(a) (b)

Fig. 2. Log-pdf of the irradiance (Subfigs. (a) and (b)) for different values of ρ, showing the transition from a K distribution ($\rho = 0$) to a gamma-gamma distribution ($\rho = 1$) using the proposed $\mathcal{M}$ distribution, in the case of strong irradiance fluctuations (a) and weak irradiance fluctuations (b). In both figures, $\Omega = 0$, $2b_0 = 1$ and $\phi_A - \phi_B = \pi/2$.

5. Comparison with experimental plane wave and spherical wave data

Flatté et al. (Flatté et al., 1994) calculated the pdf from numerical simulations for a plane wave propagated through homogeneous and isotropic atmospheric turbulence and compared the results with several pdf models. On the other hand, Hill et al. (Hill & Frehlich, 1997) used numerical simulation of the propagation of a spherical wave through homogeneous and isotropic turbulence that also led to pdf data for the log-irradiance fluctuations. In this section, we compare our $\mathcal{M}$ distribution model with some of the published numerical simulation data plots in (Flatté et al., 1994) and (Hill & Frehlich, 1997) of the log-irradiance pdf, covering a range of conditions that extends from weak irradiance fluctuations far into the saturation regime characterized by a Rytov variance, σ_1^2, of 25, where σ_1^2=$1.23C_n^2k^{7/6}L^{11/6}$, In that expression, k=$2\pi/\lambda$ is the optical wave number, λ is the wavelength, C_n^2 is the atmospheric refractive-index structure parameter and L is the propagation path length between transmitter and receiver. For values less than unity, the Rytov variance is the scintillation index (normalized variance of irradiance) of a plane wave in the absence of inner scale effects and for values greater than unity it is considered a measure of the strength of optical fluctuations. The $\mathcal{M}$ distribution model employed in this section to fit with the experimental numerical data is intentionally restricted to have its β parameter as a natural number in all cases. Hence, the infinite summation included in the closed form expression obtained for the generalized $\mathcal{M}$ distribution (Eq. (22)) can be avoided. This fact let us offer an even more evident analytical tractability by directly employing Eq. (24), with a finite summation of β terms, and maintaining an extremely high accuracy.

For the current case of a plane wave propagated through turbulent atmosphere the simulation parameters that determine the physical situation are only l_0/R_F and σ_1^2, as explained in (Flatté et al., 1994), where l_0 is the inner scale of turbulence. The quantity $R_F = \sqrt{L/k}$ is the scale size of the Fresnel zone.

Thus, we plot in Figs. 3 (a)-(c) the predicted log-irradiance pdf associated with the $\mathcal{M}$ distribution (black solid line) for comparison with some of the simulation data illustrated in Figs. 4, 5 and 7 of (Flatté et al., 1994). The simulation pdf values are plotted as a function of $(\ln I - < \ln I >)/\sigma$, as in (Flatté et al., 1994), where $< \ln I >$ is the mean value of the log-irradiance and $\sigma = \sqrt{\sigma^2_{\ln I}}$, the latter being the root mean square (rms) value of $\ln I$. The simulation pdf's were displayed in this fashion in the hope that it would reveal their salient features. For sake of brevity, and as representative of typical atmospheric propagation, we only use the inner scale value $l_0 = 0.5R_F$ so we can include the effect of l_0 in our results. We also plot the gamma-gamma pdf (red dashed line) obtained in (Al-Habash et al., 2001) for the sake of comparison. In Fig. 3 (a) we use a Rytov variance $\sigma_1^2 = 0.1$ corresponding to weak irradiance fluctuations, in Fig. 3 (b) we employ $\sigma_1^2 = 2$ corresponding to a regime of moderate irradiance fluctuations whereas in Fig. 3 (c), σ_1^2 was established to 25 for a particular case of strong irradiance fluctuations.

Values of the scaling parameter σ required in the plots for the $\mathcal{M}$ pdf are obtained from Andrews' development (Andrews et al., 2001) in the presence of inner scale. From such development, the model for the refractive-index spectrum, $\Phi_n(\kappa)$, used is the effective atmospheric-spectrum defined by:

$$\Phi_n(\kappa) = 0.033C_n^2\kappa^{-11/3}\left[f(\kappa, l_0)\exp\left(-\frac{\kappa^2}{\kappa_x^2}\right) + \frac{\kappa^{11/3}}{(\kappa^2+\kappa_y^2)^{11/6}}\right], \tag{46}$$

where κ is the scalar spatial wave number. In Eq. (46), the inner-scale factor, $f(\kappa l_0)$, describes the spectral bump and dissipation range at high wave numbers and, from (Andrews et al.,

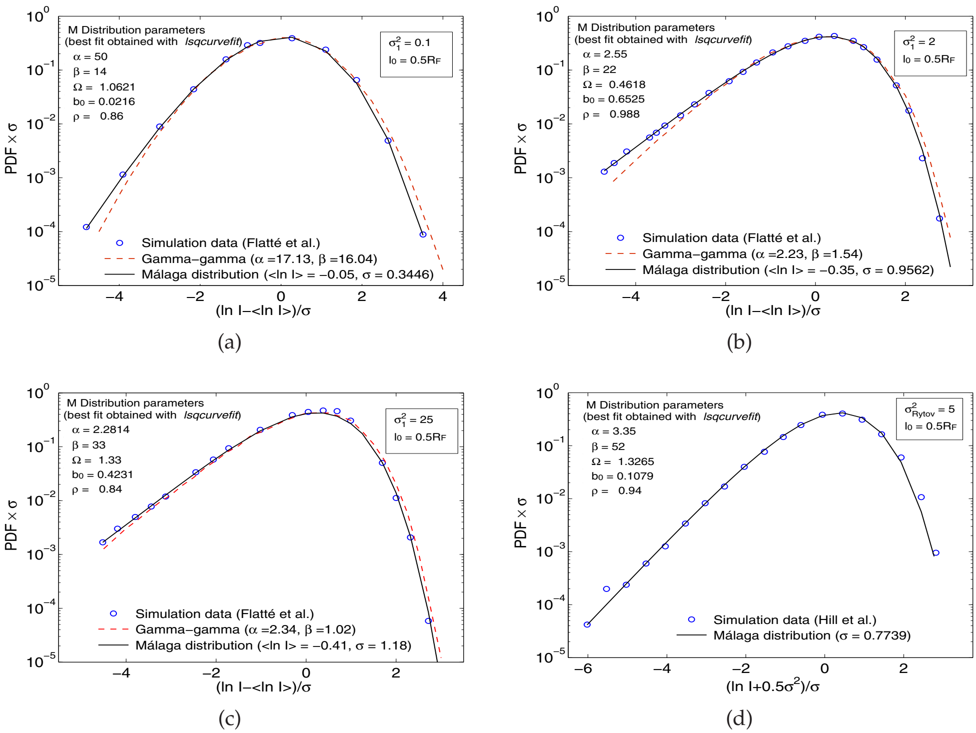


Fig. 3. The pdf of the scaled log-irradiance for a plane wave (Figures (a), (b) and (c)) and a spherical wave (Figure (d)) in the case of: (a) weak irradiance fluctuations ($\sigma_1^2 = 0.1$ and $l_0/R_F = 0.5$); (b) moderate irradiance fluctuations ($\sigma_1^2 = 2$ and $l_0/R_F = 0.5$); (c) strong irradiance fluctuations ($\sigma_1^2 = 25$ and $l_0/R_F = 0.5$); and (d) strong irradiance fluctuations ($\sigma_{Rytov}^2 = 5$ and $l_0/R_F = 0.5$). The blue open circles represent simulation data, the dashed red line is from the gamma-gamma pdf with α and β predicted in (Flatté et al., 1994) and the solid black line is from our $\mathcal{M}$ distribution model. In all subfigures, $\phi_A - \phi_B = \pi/2$.

2001), it is defined by:

$$f(\kappa, l_0) = \exp\left(-\frac{\kappa}{\kappa_l^2}\right)\left[1 + 1.802\left(\frac{\kappa}{\kappa_l}\right) - 0.254\left(\frac{\kappa}{\kappa_l}\right)^{7/6}\right], \qquad \kappa_l = \frac{3.3}{l_0}, \tag{47}$$

where it depends only on the dimensionless variable, κl_0. The limit $\kappa l_0 \to 0$ gives the inertial-range formula for $\Phi_n(\kappa)$ because $f(0) = 1$. The quantity κ_l identifies the spatial wave number associated with the inner scale, l_0 (m) of the optical turbulence. Finally, in Eq. (46), κ_x and κ_y represent cutoff spatial frequencies that eliminate mid-range scale size effects under moderate-to-strong fluctuations. Thus, if we invoke the *modified Rytov theory* then

$$\sigma = \sqrt{\sigma_{\ln_I}^2} = \sqrt{\ln\left(\sigma_I^2 + 1\right)}, \tag{48}$$

where σ_I^2 is the scintillation index. From these expressions, σ is obtained and for its calculated magnitude, the other scaling parameter, $< \ln I >$, required in the plots were directly extracted from the Figure 1 in (Flatté et al., 1994). Now, with Eq. (48) we can calculate the set of parameters $(\alpha, \beta, \gamma, \rho, \Omega^{'})$ with the constraint imposed by Eq. (27), and taking into account that we had imposed β parameter will be a natural number. Such set of parameters were obtained by running the function *lsqcurvefit* in MATLAB (Mathworks, 2011) in order to solve this nonlinear data-fitting problem. The $\mathcal{M}$ pdf curves in Figs 3 (a)-(c) provide excellent fits with the simulation data, even better than the provided by the gamma-gamma model, for all conditions of turbulence, from weak irradiance fluctuations far into the saturation regime. In particular, in Fig. 3 (a), (b) and (c) we use the simulation values $\sigma_1^2 = 0.1$, $l_0 = 0.5R_F$, $\sigma_1^2 = 2$, $l_0 = 0.5R_F$ and $\sigma_1^2 = 25$, $l_0 = 0.5R_F$ and the predicted σ from Andrews's work (Eq. (48)) is found to be $\sigma = 0.3427$, $\sigma = 0.9332$ and $\sigma = 1.0192$. The obtained values from the $\mathcal{M}$ distribution produce a "best fitting" curve with a calculated σ of: $\sigma = 0.3446$, $\sigma = 0.9562$ and $\sigma = 1.18$, respectively. Only the value obtained for $\sigma_1^2 = 25$, $l_0 = 0.5R_F$ is a bit higher than the one predicted by Andrews's work so his developments can be used as a good starting-point to obtain the set of parameters of the $\mathcal{M}$ distribution.

Finally, in Fig. 3 (d) we have obtained a very good fitting to the simulation data for a spherical wave in the case of strong irradiance fluctuations. Following Hill's representation (Hill & Frehlich, 1997), the simulation pdf data and pdf values predicted by the $\mathcal{M}$ distribution are displayed as a function of $(\ln I + 0.5\sigma^2)/\sigma$, where σ was defined in Eq. (48). In this particular case of propagating a spherical wave, various additional parameters are needed: first, the Rytov parameter, σ^2_{Rytov}, defined as the weak fluctuation scintillation index in the presence of a finite inner scale. Thus: $\sigma^2_{Rytov}=\beta_0^2\widetilde{\sigma}^2(l_0/R_F)$, as indicated in (Al-Habash et al., 2001; Andrews et al., 2001), where the quantity β_0^2 is the second additional parameter used in the analysis of the numerical simulation data for a spherical wave. Concretely, this latter parameter is the classic Rytov scintillation index of a spherical wave in the limit of weak scintillation and a Kolmogorov spectrum, defined by: $\beta_0^2 = 0.4\sigma_1^2 = 0.496C_n^2 k^{7/6} L^{11/6}$.

For the particular case displayed in Fig. 3 (d), the gamma-gamma pdf does not fit with the simulation data and, even more, the Beckmann pdf did not lend itself directly to numerical calculations and so are omitted. Nevertheless, the $\mathcal{M}$ pdf shows very good agreement with the data once again, with the advantage of a simple functional form, emphasized by the fact that its β parameter is a natural number, which leads to a closed-form representation.

6. Concluding remarks

In this chapter, a novel statistical model for atmospheric optical scintillation is presented. Unlike other models, our proposal appears to be applicable for plane and spherical waves under all conditions of turbulence from weak to super strong in the saturation regime. The proposed model unifies in a closed-form expression the existing models suggested in the bibliography for atmospheric optical communications. In addition to the mathematical expressions and developments, we have introduced a different perturbational propagation model, indicated in Fig. 1, that gives a physical sense to such existing models. Hence, the received optical intensity is due to three different contributions: first, a LOS component, second, a coupled-to-LOS scattering component, as a great novelty in the model, that includes the fraction of power traveling very closed to the line of sight, and eventually suffering from almost the same random refractive index variations than the LOS component; and third, the scattering component affected by refractive index fluctuations completely different to the other two components. The first two components are governed by a gamma distribution whereas the scattering component is depending on a circular Gaussian complex random variable. All of them let us model the amplitude of the irradiance (small-scale fluctuations), while the multiplicative perturbation that represents the large-scale fluctuations, X, and depending of the log-amplitude scintillation, χ, is approximated for a gamma distribution.

Therefore, we have derived some of the distribution models most frequently employed in the bibliography by properly choosing the magnitudes of the parameters involving the generalized $\mathcal{M}^{(G)}$ model (or, directly, $\mathcal{M}$, if β is a natural number). Then, the Rice-Nakagami distribution is obtained when U_L becomes a constant random variable while the coupled-to-LOS scattering is eliminated. As indicated in (Strohbehn, 1978), it is straightforward to obtain a lognormal distribution from this model. If we now eliminate the two components representing the scattering power, U_S^C and U_S^G, and taking again U_L as a constant, then the gamma model is derived.

To obtain the K distribution function, both the LOS component and the coupled-to-LOS scattering component must be eliminated from the model. If the effective number of discrete scatterers is unbounded then the K distribution tends to the negative exponential distribution as the gamma distribution that governs the large-scale fluctuations approaches a delta function.

To generate the gamma-gamma model, we must eliminate U_S^G. Then, this model is obtained when the LOS component and the coupled-to-LOS scattering component take part in the propagation model, i.e., the scattering contribution is, in fact, connected to the line of sight.

To close the fourth section of this chapter, we have taken the lognormal-Rician pdf as the model that provides the best fit to experimental data (Andrews et al., 2001; Churnside & Clifford, 1987). To derive such model from the $\mathcal{M}$ distribution presented in this chapter, we have suggested the gamma-Rician pdf obtained in this current work as a reasonable alternative to the LR pdf for a number of reasons. First, the gamma distribution itself has often been proposed as an approximation to the lognormal model. It is desirable to use the gamma distribution as an approximation to the lognormal pdf because of its simple functional form, which leads to a closed-form representation of the gamma-Rician pdf given by Eq. (43). This makes computations extremely easy in comparison with LR pdf. Second, parameter value α is directly related to calculated values of large-scale scintillation that depend only on values of atmospheric parameters. Third, and perhaps most important, the cumulative distribution function (cdf) for the $\mathcal{M}^{(G)}$ and the $\mathcal{M}$ pdf's can also be found in closed form, as was shown in Eqs. (28), (29). For practical purposes, it is the cdf that is of greater interest than the pdf since the former is used to predict probabilities of detection and fade in an optical communication or radar system.

Hence, knowing the physical and/or meteorological parameters of a particular link, it is at the discretion of researchers to determine, to choose or to switch among the different statistical natures offered by the closed-form analytical model presented in this work. So, in conclusion, the $\mathcal{M}$ distribution model unifies most of the proposed statistical model for the irradiance fluctuations derived in the bibliography,

Finally, we have made a number of comparisons with published plane wave and spherical wave simulation data over a wide range of turbulence conditions (weak to strong) that includes inner scale effects. The $\mathcal{M}$ distribution model is intentionally restricted to have its β parameter as a natural number for the sake of a simpler analytical tractability. The $\mathcal{M}$ distribution model is found to provide an excellent fit to the simulation data in all cases tested.

Again, we must remark that all the results shown in section 5 are obtained with β being a natural number so that the number of terms in the summation included in Eq. (24) is finite (limited, precisely, by β). This feature provides a more remarkable analytical tractability to the proposed $\mathcal{M}$ distribution that, in addition, was already written in a closed form expression.

7. Appendix A: proof of lemma 1

Starting with the pdf of the generalized distribution, $\mathcal{M}^{(G)}$, written in Eq. (22), we can proceed as follows: first, the confluent hypergeometric function of the first kind employed in Eq. (20) can be expanded by its series representation:

$$
{}_1F_1\left(a;c;z\right) = \sum_{k=1}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} \frac{z^{k-1}}{(k-1)!}, \qquad |z| < \infty; \tag{49}
$$

as indicated in (Andrews, 1998), where $(a)_k$ represents the Pochhammer symbol. Then, Eq. (20) can be expressed as:

$$
f_Y\left(y\right) = \frac{1}{\gamma}\left(\frac{\gamma\beta}{\gamma\beta+\Omega'}\right)^{\beta} \exp\left(-\frac{y}{\gamma}\right) \sum_{k=1}^{\infty} \frac{(\beta)_{k-1}}{[(k-1)!]^2} \frac{y^{k-1}\left(\Omega'\right)^{k-1}}{\gamma^{k-1}\left(\Omega'+\gamma\beta\right)^{k-1}}. \tag{50}
$$

To obtain the unconditional generalized distribution, $\mathcal{M}^{(G)}$, and from Eqs. (21) and (50), we can form:

$$
\begin{aligned}
f_I(I) = \int_0^{\infty} f_Y(I|x) f_X(x) \mathrm{d}x = \frac{1}{\gamma}\frac{\alpha^{\alpha}}{\Gamma\left(\alpha\right)}\left(\frac{\gamma\beta}{\gamma\beta+\Omega'}\right)^{\beta} \\
\sum_{k=1}^{\infty} \frac{(\beta)_{k-1}}{[(k-1)!]^2} \frac{I^{k-1}\left(\Omega'\right)^{k-1}}{\gamma^{k-1}\left(\Omega'+\gamma\beta\right)^{k-1}} \int_0^{\infty} x^{\alpha-1-k} \exp\left(-\frac{I}{\gamma x} - \alpha x\right) dx,
\end{aligned} \tag{51}
$$

having integrated term by term as the radius of convergence of Eq. (50) is infinity. Now, using Eq. (3.471-9) of Ref. (Gradshteyn & Ryzhik, 2000),

$$
\int_0^{\infty} x^{\nu-1} \exp\left(-\frac{\beta}{x} - \gamma x\right) dx = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_{\nu}\left(2\sqrt{\beta\gamma}\right), \tag{52}
$$

and substituting it into Eq. (51), we obtain:

$$
f_I\left(I\right) = \frac{1}{\gamma}\frac{\alpha^{\alpha}}{\Gamma\left(\alpha\right)}\left(\frac{\gamma\beta}{\gamma\beta+\Omega'}\right)^{\beta} \sum_{k=1}^{\infty} \frac{(\beta)_{k-1}}{[(k-1)!]^2} \frac{I^{k-1}\left(\Omega'\right)^{k-1}}{\gamma^{k-1}\left(\Omega'+\gamma\beta\right)^{k-1}} 2\left(\frac{I}{\alpha\gamma}\right)^{\frac{\alpha-k}{2}} K_{\alpha-k}\left(2\sqrt{\frac{\alpha I}{\gamma}}\right). \tag{53}
$$

Finally, Eq. (53) can be rewritten as:

$$
f_I\left(I\right) = A^{(G)} \sum_{k=1}^{\infty} a_k^{(G)} I^{\frac{\alpha+k}{2}-1} K_{\alpha-k}\left(2\sqrt{\frac{\alpha I}{\gamma}}\right), \tag{54}
$$

where

$$\begin{cases} A^{(G)} \triangleq \dfrac{2\alpha^{\frac{\alpha}{2}}}{\gamma^{1+\frac{\alpha}{2}}\Gamma(\alpha)} \left(\dfrac{\gamma\beta}{\gamma\beta+\Omega'}\right)^{\beta}; \\ a_k^{(G)} \triangleq \dfrac{(\beta)_{k-1}(\alpha\gamma)^{\frac{k}{2}}}{\left[(k-1)!\right]^2 \gamma^{k-1}(\Omega'+\gamma\beta)^{k-1}}; \end{cases} \tag{55}$$

as was already indicated in Eqs. (22) and (23).

In reference of the $\mathcal{M}(\alpha, \beta, \gamma, \rho, \Omega')$ distribution, where the β parameter represents a natural number, the way to prove the lemma is something different. In this respect, we can obtain the Laplace transform, $\mathcal{L}[f_Y(y); s]$, of the shadowed Rice single pdf, $f_Y(y)$, written in Eq. (20), in a direct way, with the help of Eq. (7) of Ref. (Abdi et al., 2003), since the moment generating function (MGF) and the Laplace transform of the pdf $f_Y(y)$ are related by $M[f_Y(y); -s] = \mathcal{L}[f_Y(y); s]$:

$$\mathcal{L}[f_Y(y); s] = \left(\frac{\gamma\beta}{\gamma\beta+\Omega'}\right)^{\beta} \frac{(1+\gamma s)^{\beta-1}}{\left(\frac{\gamma\beta}{\gamma\beta+\Omega'}+\gamma s\right)^{\beta}} = \frac{1}{\gamma}\left(\frac{\gamma\beta}{\gamma\beta+\Omega'}\right)^{\beta} \frac{\left(\frac{1}{\gamma}+s\right)^{\beta-1}}{\left(\frac{\beta}{\gamma\beta+\Omega'}+s\right)^{\beta}}. \tag{56}$$

Now, let us consider the following Laplace-transform pair

$$\Gamma(\nu+1)(s-\lambda)^n (s-\mu)^{-\nu-1} \Leftrightarrow n! t^{\nu-n} e^{\mu t} L_n^{\nu-n}\left[(\lambda-\mu)t\right], \qquad \mathrm{Re}(\nu) > n-1; \tag{57}$$

given in (Elderlyi, 1954), Eq. (4) in pp. 238, where the minor error in the sign of the argument of the Laguerre polynomial found and corrected in (Paris, 2010) has already taken into account. If we denote $\lambda = -\frac{1}{\gamma}$, $\mu = -\frac{\beta}{\gamma\beta+\Omega'}$, $n = \beta - 1$ and $\nu = \beta - 1$, then

$$(\beta-1)!\left(s+\frac{1}{\gamma}\right)^{\beta-1}\left(s+\frac{\beta}{\gamma\beta+\Omega'}\right)^{-\beta} \Leftrightarrow (\beta-1)! e^{-\frac{\beta}{\gamma\beta+\Omega'}t} L_{\beta-1}\left[\frac{-\Omega'}{\gamma\beta+\Omega'}\frac{t}{\gamma}\right], \tag{58}$$

where $L_n[\cdot]$ is the Laguerre polynomial of order n. If we substitute Eq. (58) into Eq. (56), then the pdf of Y can be expressed as:

$$f_Y(y) = \frac{1}{\gamma}\left(\frac{\gamma\beta}{\gamma\beta+\Omega'}\right)^{\beta} \exp\left(-\frac{\beta}{\gamma\beta+\Omega'}y\right) L_{\beta-1}\left[\frac{-\Omega' y}{(\gamma\beta+\Omega')\gamma}\right]. \tag{59}$$

Now, to obtain the unconditional $\mathcal{M}$ distribution, from Eqs. (21) and (59), we can form:

$$\begin{aligned} f_I(I) &= \int_0^{\infty} f_Y(I|x) f_X(x) \mathrm{d}x = \\ &= \frac{\alpha^{\alpha}}{\gamma\Gamma(\alpha)}\left(\frac{\gamma\beta}{\gamma\beta+\Omega'}\right)^{\beta} \int_0^{\infty} \frac{1}{x}\exp\left(-\frac{\beta}{\gamma\beta+\Omega'}\frac{I}{x}\right) L_{\beta-1}\left[\frac{-\Omega'}{\gamma\beta+\Omega'}\frac{I}{\gamma x}\right] x^{\alpha-1}\exp(-\alpha x)\, dx. \end{aligned} \tag{60}$$

By expressing the Laguerre polynomial in a series,

$$L_n[x] = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!}, \tag{61}$$

as was shown in Eq. (8.970-1) of Ref. (Gradshteyn & Ryzhik, 2000), it follows that Eq. (60) becomes

$$f_I(I) = \frac{\alpha^\alpha}{\Gamma(\alpha)} \frac{1}{\gamma} \left(\frac{\gamma\beta}{\gamma\beta + \Omega'} \right)^\beta \sum_{k=1}^{\beta} (-1)^{k-1} \binom{\beta-1}{k-1} \frac{1}{(k-1)!} \left(\frac{-\Omega'}{\gamma\beta + \Omega'} \frac{1}{\gamma} \right)^{k-1} I^{k-1} \cdot \int_0^\infty \exp\left(-\frac{\beta}{\gamma\beta + \Omega'} \frac{I}{x} \right) x^{\alpha-1-k} \exp(-\alpha x)\, dx. \tag{62}$$

Now, we denote by G_k the integral:

$$G_k = \int_0^\infty x^{\alpha-1-k} \exp\left(-\frac{\beta}{\gamma\beta + \Omega'} \frac{I}{x} - \alpha x \right) dx. \tag{63}$$

Again, using Eq. (52), we can solve G_k:

$$G_k = 2 \left(\frac{\beta}{\alpha(\gamma\beta + \Omega')} \right)^{\frac{\alpha-k}{2}} I^{\frac{\alpha-k}{2}} K_{\alpha-k} \left(2\sqrt{\frac{\beta I}{\gamma\beta + \Omega'} \alpha} \right). \tag{64}$$

Employing this latter result and inserting it into Eq. (62), we find the pdf of I in the form:

$$f_I(I) = A^{(G)} \left(\frac{\gamma\beta}{\gamma\beta + \Omega'} \right)^{\frac{\alpha}{2}} \sum_{k=1}^{\beta} a_k I^{\frac{\alpha+k}{2}-1} K_{\alpha-k} \left(2\sqrt{\frac{\alpha\beta I}{\gamma\beta + \Omega'}} \right), \tag{65}$$

where, again, we can identify $A^{(G)}$ and a_k parameters as the ones given by Eq. (25). □

8. Appendix B: proof of lemma 2

As indicated in Eq. (18), the observed irradiance, I, of our proposed propagation model can be expressed as: $I = XY$, where the pdf of variables X and Y were written in Eqs. (21) and (20), respectively. Based on assumptions of statistical independence for the underlying random processes, X and Y, then:

$$m_k(I) = E\left[X^k\right] E\left[Y^k\right] = m_k(X)\, m_k(Y). \tag{66}$$

From Eq. (2.23) of Ref. (Simon & Alouini, 2005), the moment of a Nakagami-m pdf is given by:

$$m_k(X) = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)\, \alpha^k}; \tag{67}$$

and, from Eq. (2.69) of Ref. (Simon & Alouini, 2005), the moment of the Rician-shadowed distribution is given by:

$$m_k(Y) = \left(\frac{\gamma\beta}{\gamma\beta + \Omega'} \right)^\beta \gamma^k \Gamma(k+1)\, {}_2F_1\left(k+1, \beta; 1; \frac{\Omega'}{\gamma\beta + \Omega'} \right). \tag{68}$$

When performing the product of Eq. (67) by Eq. (68), we finally obtain the centered moments for the generalized distribution, $\mathcal{M}^{(G)}$, as was written in Eq. (26).

On the other hand, in reference to the $\mathcal{M}(\alpha,\beta,\gamma,\rho,\Omega^{'})$ distribution, where the β parameter is restricted to be a natural number for this particular case, we can proceed as follows: from Eq. (59), we can obtain the moment of the Rician-shadowed distribution, given by:

$$\begin{aligned} m_k(Y) &= \frac{1}{\gamma}\left(\frac{\gamma\beta}{\gamma\beta+\Omega^{'}}\right)^{\beta}\int_0^{\infty} y^k \exp\left(-\frac{\beta y}{\gamma\beta+\Omega^{'}}\right)L_{\beta-1}\left[\frac{-\Omega^{'}y}{\gamma\left(\gamma\beta+\Omega^{'}\right)}\right]dy \\ &= \frac{1}{\gamma}\left(\frac{\gamma\beta}{\gamma\beta+\Omega^{'}}\right)^{\beta}\sum_{r=0}^{\beta-1}\binom{\beta-1}{r}\frac{1}{r!}\left(\frac{\Omega^{'}}{\gamma\left(\gamma\beta+\Omega^{'}\right)}\right)^{r}\int_0^{\infty} y^{k+r}\exp\left(-\frac{\beta y}{\gamma\beta+\Omega^{'}}\right)dy. \end{aligned} \tag{69}$$

Now, from Eq. (3.381-4) of Ref. (Gradshteyn & Ryzhik, 2000),

$$\int_0^{\infty} x^{\nu-1}\exp(-\mu x)dx = \frac{1}{\mu^{\nu}}\Gamma(\nu), \qquad [\mathrm{Re}(\mu)>0,\quad \mathrm{Re}(\nu)>0], \tag{70}$$

we can express Eq. (69) as:

$$m_k(Y) = \frac{1}{\gamma}\left(\frac{\gamma\beta}{\gamma\beta+\Omega^{'}}\right)^{\beta}\sum_{r=0}^{\beta-1}\binom{\beta-1}{r}\frac{1}{r!}\left(\frac{\Omega^{'}}{\gamma\left(\gamma\beta+\Omega^{'}\right)}\right)^{r}\frac{\Gamma(k+r+1)}{\left(\frac{\beta}{\gamma\beta+\Omega^{'}}\right)^{k+r+1}}. \tag{71}$$

Finally, when performing the product of Eq. (67) by Eq. (71), we certainly obtain Eq. (27). □

9. Appendix C: proof of lemma 3

For both cases, when I follows a generalized distribution, $\mathcal{M}^{(G)}$, or directly an $\mathcal{M}$ distribution if its β parameter is a natural number, then we need to solve the same integral. Thus, from Eq. (6.592.2) of Ref. (Gradshteyn & Ryzhik, 2000),

$$\begin{aligned} &\int_0^{1} x^{\lambda}(1-x)^{\mu-1}K_{\nu}\left(a\sqrt{x}\right)dx = \\ &2^{-\nu-1}a^{-\nu}\frac{\Gamma(\nu)\Gamma(\mu)\Gamma\left(\lambda+1-\frac{1}{2}\nu\right)}{\Gamma\left(\lambda+1+\mu-\frac{1}{2}\nu\right)}\,{}_1F_2\left(\lambda+1-\frac{1}{2}\nu;1-\nu,\lambda+1+\mu-\frac{1}{2}\nu;\frac{a^2}{4}\right)+ \\ &+2^{-1-\nu}a^{\nu}\frac{\Gamma(-\nu)\Gamma\left(\lambda+1+\frac{1}{2}\nu\right)\Gamma(\mu)}{\Gamma\left(\lambda+1+\mu+\frac{1}{2}\nu\right)}\,{}_1F_2\left(\lambda+1+\frac{1}{2}\nu;1+\nu,\lambda+1+\mu+\frac{1}{2}\nu;\frac{a^2}{4}\right), \\ &\qquad \mathrm{Re}(\lambda) > -1+\frac{1}{2}|\mathrm{Re}(\nu)|, \qquad \mathrm{Re}(\mu)>0; \end{aligned} \tag{72}$$

and by making the following change of variables: $x' = I_T \cdot x$, then $\mathrm{d}x' = I_T\mathrm{d}x$; and identifying $\mu = 1$, $\lambda = (\alpha+k)/2$, $\nu = \alpha - k$, where $a = 2\sqrt{\alpha/(\gamma I_T)}$ and $a = 2\sqrt{\alpha\beta/([\gamma\beta+\Omega']I_T)}$ for the $\mathcal{M}^{(G)}$ and the $\mathcal{M}$ distribution, respectively; thus, the cdf associated with the $\mathcal{M}^{(G)}$ and the $\mathcal{M}$ distribution is readily found to be the expressions indicated in Eqs. (28) and (29). □

10. Acknowledgment

This work was fully supported by the Spanish Ministerio de Ciencia e Innovación, Project TEC2008-06598.

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