SINGULARITY OF FRACTAL MEASURE ASSOCIATED WITH THE $(0,1,7)$ - PROBLEM

Truong Thi Thuy Duong

Vinh University, Nghe An

Vu Hong Thanh

Pedagogical College of Nghe An

Abstract. Let μ be the probability measure induced by $S = \sum_{n=1}^{\infty} 3^{-n} X_n$, where $X_1, X_2, ...$ is a sequence of independent, identically distributed (i.i.d) random variables each taking values $0, 1, a$ with equal probability $1/3$. Let $\alpha(s)$ (resp.$\underline{\alpha}(s), \overline{\alpha}(s)$) denote the local dimension (resp. lower, upper local dimension) of $s \in$ supp μ, and let

$$\overline{\alpha} = \sup\{\overline{\alpha}(s) : s \in \text{supp } \mu\}; \underline{\alpha} = \inf\{\underline{\alpha}(s) : s \in \text{supp } \mu\};$$
$$E = \{\alpha : \alpha(s) = \alpha \text{ for some } s \in \text{supp } \mu\}$$

.

In the case $a = 3k + 1$ for $k = 1$, $E = [1 - \frac{\log(1+\sqrt{5}) - \log 2}{\log 3}, 1]$, see [10]. It is conjectured that in the general case, for $a = 3k + 1$ ($k \in \mathbb{N}$), the local dimension is of the form as the case $k = 1$, i.e., $E = [1 - \frac{\log a}{b \log 3}, 1]$ for a, b depends on k. In fact, our result shows that for $k = 2$ $(a = 7)$, we have $\overline{\alpha} = 1, \underline{\alpha} = 1 - \frac{\log(1+\sqrt{3})}{3 \log 3}$ and $E = [1 - \frac{\log(1+\sqrt{3})}{3 \log 3}, 1]$.

1. Introduction

Let X be random variable taking values $a_1, a_2, ..., a_m$ with probability $p_1, p_2, ..., p_m$, respectively and let $X_1, X_2, ...$ be a sequence of independent random variables with the same distribution as X. Let $S = \sum_{n=1}^{\infty} \rho^n X_n$, for $0 < \rho < 1$, and let μ be the probability measure induced by S, i.e.,

$$\mu(A) = \text{Prob}\{\omega : S(\omega) \in A\}.$$

It is known that the measure is either purely singular or absolutely continuous.

An intriguing case when $m = 3, \rho = p_1 = p_2 = p_3 = 1/3$ and $a_1 = 0, a_2 = 1, a_3 = a$. According to the "pure theorem" of Lagarias and Wang, in [7], if $a \equiv 0 \pmod 3$ or $a \equiv 1 \pmod 3$ then μ is purely singular.

Let us recall that for $s \in$ supp μ the *local dimension* $\alpha(s)$ of μ at s is defined by

$$\alpha(s) = \lim_{h \to 0^+} \frac{\log \mu(B_h(s))}{\log h}, \tag{1}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

provided that the limit exists, where $B_h(s)$ denotes the ball centered at s with radius h. If the limit (1) does not exist, we define the upper and lower local dimension, denoted $\overline{\alpha}(s)$ and $\underline{\alpha}(s)$, by taking the upper and lower limits respectively.

Denote

$$\overline{\alpha} = \sup\{\overline{\alpha}(s) : s \in \text{supp } \mu\} \ ; \ \underline{\alpha} = \inf\{\underline{\alpha}(s) : s \in \text{supp } \mu\};$$

and let

$$E = \{\alpha : \alpha(s) = \alpha \text{ for some } s \in \text{supp } \mu\}$$

be the attainable values of $\alpha(s)$, i.e., the range of function α definning in the supp μ.

In the case $a = 3k + 1$ it is conjectured that the local dimension is of the form as $k = 1$, it means that $E = [1 - \frac{\log a}{b \log 3}, 1]$ for a, b depends on k. Our aim in this note is to prove that this conjecture is true for $k = 2$. In fact, our result is the following:

Main Theorem. For $a = 7$ we have $\overline{\alpha} = 1, \underline{\alpha} = 1 - \frac{\log(1+\sqrt{3})}{3 \log 3}$ and $E = [1 - \frac{\log(1+\sqrt{3})}{3 \log 3}, 1]$. The paper is organized as follows. In Section 2 we establish some auxiliary results used in the proof of the Main Theorem. The proof of the Main Theorem will be given in the last section.

2. Some Auxiliary Results

Let $X_1, X_2, \dots$ be a sequence of i.i.d random variables each taking values $0, 1, 7$ with equal probability 1/3. Let $S = \sum_{i=1}^{\infty} 3^{-i} X_i$, $S_n = \sum_{i=1}^{n} 3^{-i} X_i$ be the n-partial sum of S, and let μ, μ_n be the probability measures induced by S, S_n, respectively. For any $s = \sum_{n=1}^{\infty} 3^{-n} x_n \in$ supp μ, $x_n \in$ D: $= \{0, 1, 7\}$, let $s_n = \sum_{i=1}^{n} 3^{-i} x_i$ be its n-partial sum. It is easy to see that for any $s_n, s'_n \in$ supp $\mu_n, |s_n - s'_n| = k3^{-n}$ for some $k \in \mathbb{N}$.

Let

$$\langle s_n \rangle = \{(x_1, x_2, ..., x_n) \in D^n : \sum_{i=1}^{n} 3^{-i} x_i = s_n\}.$$

Then we have

$$\mu_n(s_n) = \#\langle s_n \rangle 3^{-n} \text{ for every } n, \tag{2}$$

where $\#\langle s_n \rangle$ denotes the cardinality of set $\langle s_n \rangle$.

Two sequences $(x_1, x_2, ..., x_n)$ and $(x'_1, x'_2, ..., x'_n)$ in D^n are said to be *equivalent*, denoted by $(x_1, x_2, \dots, x_n) \approx (x'_1, x'_2, \dots, x'_n)$, if $\sum_{i=1}^{n} 3^{-i} x_i = \sum_{i=1}^{n} 3^{-i} x'_i$. We have

2.1. Claim. (i) For any $(x_1, x_2, \dots, x_n)$, $(x'_1, x'_2, \dots, x'_n)$ in D^n and $s_n = \sum_{i=1}^{n} 3^{-i} x_i$, $s'_n = \sum_{i=1}^{n} 3^{-i} x'_i$. If $s_n - s'_n = \frac{k}{3^n}$, where $k \in Z$, then $x_n - x'_n \equiv k \pmod 3$.

(ii) Let $s_n > s'_n > s''_n$ be three arbitrary consecutive points in supp μ_n. Then either $s_n - s'_n$ or $s'_n - s''_n$ is not $\frac{1}{3^n}$ and either $s_n - s'_n$ or $s'_n - s''_n$ is not $\frac{2}{3^n}$.

Proof. (i) Since $s_n - s'_n = \frac{k}{3^n}$, we have

$$3^{n-1}(x_1 - x'_1) + 3^{n-2}(x_2 - x'_2) + \dots + 3(x_{n-1} - x'_{n-1}) + x_n - x'_n = k,$$

which implies $x_n - x'_n \equiv k \pmod 3$. The claim (i) is proved.

(ii) We can write

$$s_n = s_{n-1} + \frac{x_n}{3^n},\ s'_n = s'_{n-1} + \frac{x'_n}{3^n} \text{ and } s''_n = s''_{n-1} + \frac{x''_n}{3^n},$$

where $s_{n-1}, s'_{n-1}, s''_{n-1} \in \text{supp } \mu_{n-1}$ and $x_n, x'_n, x''_n \in D$. Assume on the contrary that $s_n - s'_n = s'_n - s''_n = \frac{1}{3^n}$. Then

$$\begin{aligned} s_{n-1} - s'_{n-1} &= \frac{1 + x'_n - x_n}{3^n} = \frac{1 + x'_n - x_n}{3}\frac{1}{3^{n-1}}, \\ s'_{n-1} - s''_{n-1} &= \frac{1 + x''_n - x'_n}{3^n} = \frac{1 + x''_n - x'_n}{3}\frac{1}{3^{n-1}}. \end{aligned}$$

Since $s_k - s'_k = \frac{t}{3^k}, t \in \mathbb{Z}$ whenever $s_k, s'_k \in \text{supp } \mu_k$, we have $(1 + x'_n - x_n) \equiv 0 \pmod 3$ and $(1 + x''_n - x'_n) \equiv 0 \pmod 3$. Because $(1 + x'_n - x_n) \equiv 0 \pmod 3$ and $x_n, x'_n \in D$, we obtain $x'_n = 0$. Then $1 + x''_n - x'_n = 1 + x''_n \in \{1, 2, 8\}$ for any $x''_n \in D$, a contradiction. Similarly, we have either $s_n - s'_n$ or $s'_n - s''_n$ is not $\frac{2}{3^n}$. The claim (ii) is proved.

2.2. Claim. (i) Let $s_{n+1} \in$ supp μ_{n+1} and $s_{n+1} = s_n + \frac{0}{3^{n+1}}, s_n \in \text{supp } \mu_n$. We have

$$\#\langle s_{n+1}\rangle = \#\langle s_n\rangle, \text{for every } n \geqslant 1.$$

(ii) For any $s_n, s'_n \in \text{supp } \mu_n$ if $s_n - s'_n = \frac{1}{3^n}$, then $x'_n = 0, x_n = 1$ or $x_n = 7$ and $\#\langle s'_n\rangle \leqslant \#\langle s_n\rangle$. If $x_n = 1$, then $s_{n-1} = s'_{n-1}$ and if $x_n = 7$, then $s'_{n-1} = s_{n-1} + \frac{2}{3^{n-1}}$, where $s_n = s_{n-1} + \frac{x_n}{3^n},\ s'_n = s'_{n-1} + \frac{x'_n}{3^n}$ and $s_{n-1}, s'_{n-1} \in \text{supp } \mu_{n-1}, x_n, x'_n \in D$.

(iii) For any $s_n, s'_n \in \text{supp } \mu_n$, if $s_n - s'_n = \frac{2}{3^n}$, then $x_n = 0, x'_n = 1$ or $x'_n = 7$. Moreover, if $x'_n = 1$, then $s_{n-1} - s'_{n-1} = \frac{1}{3^{n-1}}$ and if $x'_n = 7$, then $s_{n-1} - s'_{n-1} = \frac{3}{3^{n-1}}$, where $s_n = s_{n-1} + \frac{x_n}{3^n},\ s'_n = s'_{n-1} + \frac{x'_n}{3^n}$ and $s_{n-1}, s'_{n-1} \in \text{supp } \mu_{n-1}, x_n, x'_n \in D$.

(iv) For any $s_n, s'_n, s''_n \in \text{supp } \mu_n$, if $s_n - s'_n = \frac{1}{3^n}$ and $s'_n - s''_n = \frac{2}{3^n}$, then $s_n = s_{n-1} + \frac{1}{3^n} = s^*_{n-1} + \frac{7}{3^n}$ and $s''_n = s''_{n-1} + \frac{7}{3^n}$ or $s''_n = s''_{n-1} + \frac{1}{3^n} = s'''_{n-1} + \frac{7}{3^n}$ and $s_n = s_{n-1} + \frac{1}{3^n}$, where $s_{n-1}, s''_{n-1}, s^*_{n-1}, s'''_{n-1} \in \text{supp } \mu_{n-1}$.

Proof. (i) It follows directly from Claim 2.1.

(ii) Since $s_n - s'_n = \frac{1}{3^n}$, by Claim 2.1 (i) $x_n - x'_n \equiv 1 \pmod 3$. Then $x'_n = 0, x_n = 1$ or $x_n = 7$. Therefore $s'_n = s'_{n-1} + \frac{0}{3^n}$. By Claim 2.1 (i) we have $\#\langle s'_n\rangle = \#\langle s'_{n-1}\rangle$ and $s_n = s_{n-1} + \frac{1}{3^n} = s'_{n-1} + \frac{1}{3^n}$. If s_n has an other representation $s_n = s^*_{n-1} + \frac{7}{3^n}$, then $\#\langle s_n\rangle \geqslant \#\langle s'_{n-1}\rangle = \#\langle s'_n\rangle$. If $x_n = 1$, then $s_{n-1} - s'_{n-1} = s'_{n-1} - s'_{n-1} = 0$. If $x_n = 7$, then $s_n = s^*_{n-1} + \frac{7}{3^n}, s'_n = s'_{n-1} + \frac{0}{3^n}$. It implies

$$\frac{1}{3^n} = s_n - s'_n = s^*_{n-1} - s'_{n-1} + \frac{7}{3^n}.$$

Therefore

$$s'_{n-1} - s^*_{n-1} = \frac{2}{3^{n-1}}.$$

(iii) It is proved similarly to Claim 2.2 (ii).

(iv) Since $s_n - s'_n = \frac{1}{3^n}$, by Claim 2.2 (ii) we have

$$s'_n = s'_{n-1} + \frac{0}{3^n}, s_n = s'_{n-1} + \frac{1}{3^n} = s^*_{n-1} + \frac{7}{3^n}.$$

On the other hand

$$s'_n - s''_n = \frac{2}{3^n},$$

so if

$$s''_n = s''_{n-1} + \frac{7}{3^n} = s'''_{n-1} + \frac{1}{3^n},$$

then

$$s'_{n-1} - s'''_{n-1} = s'''_{n-1} - s^*_{n-1} = \frac{1}{3^{n-1}},$$

a contradiction to Claim 2.1 (ii). Therefore $s''_n = s''_{n-1} + \frac{7}{3^n}$. Similarly, we get the last assertion.

Remark 1. 1) By Claim 2.1 (i), it follows that if $s_{n+1} \in \text{supp } \mu_{n+1}$ and $s_{n+1} = s_n + \frac{0}{3^{n+1}}$, then it can not be represented in the forms

$$s_{n+1} = s'_n + \frac{1}{3^{n+1}}, \text{ or } s_{n+1} = s''_n + \frac{7}{3^{n+1}},$$

where $s_n, s'_n, s''_n \in \text{supp } \mu_n$. Thus, any $s_{n+1} \in \text{supp } \mu_{n+1}$ has at most two representations through points in supp μ_n.

2) Assume that $s_n, s'_n \in \text{supp } \mu_n$, if $s_n - s'_n = \frac{1}{3^n}$ or $s_n - s'_n = \frac{2}{3^n}$, then s_n, s'_n are two consecutive points in supp μ_n.

2.3. Lemma. *For any two consecutive points s_n and s'_n in supp μ_n, we have*

$$\frac{\mu_n(s_n)}{\mu_n(s'_n)} \leqslant n.$$

Proof. By (2) it is sufficient to show that $\frac{\#\langle s_n \rangle}{\#\langle s'_n \rangle} \leqslant n$. We will prove the inequality by induction. Clearly the inequality holds for $n = 1$. Suppose that it is true for all $n \leqslant k$. Let $s_{k+1} > s'_{k+1}$ be two arbitrary consecutive points in supp μ_{k+1}. Write

$$s_{k+1} = s_k + \frac{x_{k+1}}{3^{k+1}}, \ s_k \in \text{supp } \mu_k, x_{k+1} \in D.$$

We consider the following cases for x_{k+1}.

Case 1. If $x_{k+1} = 7$ then $s_{k+1} = s_k + \frac{7}{3^{k+1}}$. Assume that s_{k+1} has an other representation $s_{k+1} = s^*_k + \frac{x_{k+1}}{3^{k+1}}$, $x_{k+1} \in D$. Then $s^*_k > s_k$, where $s^*_k \in \text{supp } \mu_k$.

Let $s'_k \in \text{supp } \mu_k$ be the smallest value larger than s_k. Then $s'_k > s_k$ are two consecutive points in supp μ_k.

a) For $s'_k = s_k + \frac{1}{3^k}$. If $s''_k \geqslant s'_k$ then $s''_k - s_k \geqslant \frac{3}{3^k} = \frac{9}{3^{k+1}}$. We have

$$s_{k+1} = s_k + \frac{7}{3^{k+1}} < s_k + \frac{9}{3^{k+1}} \leqslant s''_k,$$

so s_{k+1} has the unique representation through point s_k in supp μ_k. Hence $\#\langle s_{k+1}\rangle = \#\langle s_k\rangle$.

Since $s_{k+1} > s'_{k+1}$ are two arbitrary consecutive points in supp μ_{k+1} and $s_{k+1} = s_k + \frac{7}{3^{k+1}} = s'_k + \frac{4}{3^{k+1}}$, we have $s'_{k+1} = s'_k + \frac{1}{3^{k+1}}$. Assume that s'_{k+1} has an other representation $s'_{k+1} = s'''_k + \frac{7}{3^{k+1}}$. Then $s'_k - s'''_k = \frac{2}{3^k}$. It implies $s'_k - s_k = s_k - s'''_k = \frac{1}{3^k}$, which contradicts to Claim 2.1 (ii). Hence $\#\langle s'_{k+1}\rangle = \#\langle s'_k\rangle$. Therefore

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} = \frac{\#\langle s_k\rangle}{\#\langle s'_k\rangle} \leqslant k < k+1.$$

b) For $s'_k = s_k + \frac{2}{3^k} = s_k + \frac{6}{3^{k+1}}$. We have

$$s_{k+1} = s_k + \frac{7}{3^{k+1}} = s'_k + \frac{1}{3^{k+1}}.$$

It follows that

$$\#\langle s_{k+1}\rangle = \#\langle s_k\rangle + \#\langle s'_k\rangle \text{ and } \ s'_{k+1} = s'_k + \frac{0}{3^{k+1}}.$$

Hence $\#\langle s'_{k+1}\rangle = \#\langle s'_k\rangle$. Therefore

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} = \frac{\#\langle s_k\rangle + \#\langle s'_k\rangle}{\#\langle s_k\rangle} \leqslant k+1.$$

c) For $s'_k \geqslant s_k + \frac{3}{3^k} = s_k + \frac{9}{3^{k+1}}$. We have

$$s_{k+1} = s_k + \frac{7}{3^{k+1}} < s_k + \frac{9}{3^{k+1}} \leqslant s'_k,$$

so s_{k+1} has the unique representation through point s_k in supp μ_k. Hence $\#\langle s_{k+1}\rangle = \#\langle s_k\rangle$.

Since $s_{k+1} > s'_{k+1}$ are two consecutive points in supp μ_{k+1} and $s_{k+1} = s_k + \frac{7}{3^{k+1}} < s_k + \frac{9}{3^{k+1}} \leqslant s'_k$, s'_{k+1} only represents through points not bigger then s_k in supp μ_k. Let $s^*_k < s_k$ be the consecutive point for s_k in supp μ_k. We consider following three cases.
c1) If $s_k = s^*_k + \frac{1}{3^k}$, then $s'_{k+1} = s^*_k + \frac{7}{3^{k+1}}$ is the unique representation through point s^*_k in supp μ_k. It implies $\#\langle s'_{k+1}\rangle = \#\langle s^*_k\rangle$. Therefore

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} = \frac{\#\langle s_k\rangle}{\#\langle s^*_k\rangle} \leqslant k < k+1.$$

c2) If $s_k = s^*_k + \frac{2}{3^k}$, then $s'_{k+1} = s^*_k + \frac{7}{3^{k+1}} = s_k + \frac{1}{3^{k+1}}$. So

$$\#\langle s'_{k+1}\rangle = \#\langle s^*_k\rangle + \#\langle s_k\rangle.$$

Therefore
$$\frac{\#\langle s'_{k+1}\rangle}{\#\langle s_{k+1}\rangle} = \frac{\#\langle s^*_k\rangle + \#\langle s_k\rangle}{\#\langle s_k\rangle} \leqslant k+1.$$

c3) If $s_k \geqslant s^*_k + \frac{3}{3^k}$, then $s'_{k+1} = s_k + \frac{1}{3^{k+1}}$ is the unique representation through point s_k in supp μ_k. Hence $\#\langle s'_{k+1}\rangle = \#\langle s_k\rangle$. Therefore

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} = \frac{\#\langle s_k\rangle}{\#\langle s_k\rangle} = 1 < k+1.$$

Case 2. If $x_{k+1} = 0$, then $s_{k+1} = s_k + \frac{0}{3^{k+1}}$. By Claim 2.2 (i), we have $\#\langle s_{k+1}\rangle = \#\langle s_k\rangle$. Then $s'_{k+1} = s^*_k + \frac{x^*_{k+1}}{3^{k+1}} < s_{k+1} = s_k$. It implies $s^*_k < s_k$.
Let $s'_k \in$ supp μ_k be the biggest value smaller than s_k. Then $s'_k < s_k$ are two consecutive points in supp μ_k.

a) If $s_k = s'_k + \frac{1}{3^k}$, then by Claim 2.2 (ii)

$$\#\langle s'_k\rangle \leqslant \#\langle s_k\rangle. \tag{3}$$

We have
$$s_{k+1} = s_k = s'_k + \frac{3}{3^{k+1}} > s'_k + \frac{1}{3^{k+1}}.$$
Hence
$$s'_{k+1} = s'_k + \frac{1}{3^{k+1}} \geqslant s''_k + \frac{7}{3^{k+1}},$$
where $s'_k > s''_k$ are two consecutive points in supp μ_k(Because by Claim 2.1, $s_k - s'_k \geqslant \frac{2}{3^k}$). Thus, $\#\langle s'_{k+1}\rangle \leqslant \#\langle s'_k\rangle + \#\langle s''_k\rangle$. Therefore

$$\frac{\#\langle s'_{k+1}\rangle}{\#\langle s_{k+1}\rangle} \leqslant \frac{\#\langle s'_k\rangle + \#\langle s''_k\rangle}{\#\langle s_k\rangle} \leqslant \frac{\#\langle s'_k\rangle + \#\langle s''_k\rangle}{\#\langle s'_k\rangle} \leqslant (k+1).$$

b) If $s_k = s'_k + \frac{2}{3^k}$, then

$$s_{k+1} = s_k = s'_k + \frac{2}{3^k} = s'_k + \frac{6}{3^{k+1}} \geqslant s''_k + \frac{1}{3^k} + \frac{6}{3^{k+1}} = s''_k + \frac{9}{3^{k+1}},$$

with $s'_k > s''_k$ are two consecutive points in supp μ_k and $s'_k - s''_k = \frac{1}{3^k}$ or $s'_k - s''_k \geqslant \frac{3}{3^k}$.
b1) If $s'_k = s''_k + \frac{1}{3^k}$, then $\#\langle s''_k\rangle \leqslant \#\langle s'_k\rangle$, $s'_{k+1} = s''_k + \frac{7}{3^{k+1}}$ and it is the unique representation of s'_{k+1} through points in supp μ_k. (If it is not the case, $s'_{k+1} = s^*_k + \frac{1}{3^{k+1}}$, then $s_k - s^*_k = s^*_k - s'_k = \frac{1}{3^k}$, a contradictions to Claim 2.1). Hence $\#\langle s'_{k+1}\rangle = \#\langle s''_k\rangle$. Therefore

$$\frac{\#\langle s'_{k+1}\rangle}{\#\langle s_{k+1}\rangle} = \frac{\#\langle s''_k\rangle}{\#\langle s_k\rangle} \leqslant \frac{\#\langle s'_k\rangle}{\#\langle s_k\rangle} \leqslant k < k+1.$$

To show that $\frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} \leqslant k+1$, we note that

$$s_{k+1} = s_k = s_{k-1} + \frac{x_k}{3^k},\ s''_k = s''_{k-1} + \frac{x''_k}{3^k}.$$

Since $s'_k - s''_k = \frac{1}{3^k}$, we have $x''_k = 0$ and $\#\langle s''_k\rangle = \#\langle s''_{k-1}\rangle$.

Since $s_k - s''_k = \frac{3}{3^k}$, we have $x_k - x''_k \equiv 0 \pmod 3$. By $x''_k = 0$, we get $x_k = 0$. It implies $\#\langle s_k\rangle = \#\langle s_{k-1}\rangle$. Moreover, $s_{k-1} - s''_{k-1} = \frac{1}{3^{k-1}}$, so s_{k-1}, s''_{k-1} are two consecutive points in supp μ_{k-1}. Therefore, by the inductive hypothesis we have

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} = \frac{\#\langle s_k\rangle}{\#\langle s''_k\rangle} = \frac{\#\langle s_{k-1}\rangle}{\#\langle s''_{k-1}\rangle} \leqslant k-1 < k+1.$$

b2) If $s'_k \geqslant s''_k + \frac{3}{3^k}$ then

$$s_{k+1} = s_k = s'_k + \frac{6}{3^{k+1}} \geqslant s'_k + \frac{1}{3^{k+1}} > s''_k + \frac{10}{3^{k+1}}.$$

Hence $s'_{k+1} = s'_k + \frac{1}{3^{k+1}}$ and this is the unique representation of s'_{k+1} through points in supp μ_k. Hence $\#\langle s'_{k+1}\rangle = \#\langle s'_k\rangle$. Therefore

$$\frac{\#\langle s'_{k+1}\rangle}{\#\langle s_{k+1}\rangle} = \frac{\#\langle s'_k\rangle}{\#\langle s_k\rangle} \leqslant k < k+1.$$

c) If $s_k \geqslant s'_k + \frac{3}{3^k} = s'_k + \frac{9}{3^{k+1}}$, then $s'_{k+1} = s'_k + \frac{7}{3^{k+1}}$ is the unique representation of s'_{k+1}. So $\#\langle s'_{k+1}\rangle = \#\langle s'_k\rangle$. Therefore

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} = \frac{\#\langle s_k\rangle}{\#\langle s'_k\rangle} \leqslant k < k+1.$$

Case 3. If $x_{k+1} = 1$, then $s_{k+1} = s_k + \frac{1}{3^{k+1}}$. Note that if s_{k+1} has an other representation then $s_{k+1} = s^*_k + \frac{7}{3^{k+1}}$ and $s_k - s^*_k = \frac{2}{3^k}$. It implies s^*_k, s_k are two consecutive points in supp μ_k. Clearly $\#\langle s_{k+1}\rangle \leqslant \#\langle s_k\rangle + \#\langle s^*_k\rangle$. Since $s_{k+1} > s'_{k+1}$ are two arbitrary consecutive points in supp μ_{k+1}, we have $s'_{k+1} = s_k + \frac{0}{3^{k+1}}$. Hence $\#\langle s'_{k+1}\rangle = \#\langle s_k\rangle$. Therefore

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} \leqslant \frac{\#\langle s_k\rangle + \#\langle s^*_k\rangle}{\#\langle s_k\rangle} \leqslant k+1.$$

The lemma is proved.

The following proposition provides a useful formula for calculating the local dimension and it is proved similarly to the proof of Proposition 2.3 in [10] and using Lemma 2.3.

2.4. Proposition. *For $s \in$ supp μ, we have*

$$\alpha(s) = \lim_{n\to\infty} \frac{|\log \mu_n(s_n)|}{n \log 3},$$

provided that the limit exists. Otherwise, by taking the upper and lower limits respectively we get the formulas for $\overline{\alpha}(s)$ and $\underline{\alpha}(s)$.

For each infinite sequence $x = (x_1, x_2, \dots) \in D^\infty$ defines a point $s \in \text{supp}\ \mu$ by

$$s = S(x) := \sum_{n=1}^{\infty} 3^{-n} x_n.$$

We denote

$$\langle x(k) \rangle = \{(y_1, \dots, y_k) \in D^k : (y_1, \dots, y_k) \approx (x_1, \dots, x_k)\},$$

where $x(k) = (x_1, \dots, x_k)$. It is easy to check that

$$(1,0,1) \approx (0,1,7),\ (0,7,0,1) \approx (1,1,7,7) \text{ and } (1,a,0,1) \approx (0,a,7,7)$$

for any $a \in D$. We call each element in the set

$$\{(1,0,1),\ (0,1,7),\ (0,7,0,1),\ (1,1,7,7),\ (1,a,0,1),\ (0,a,7,7)\}$$

a *generator.*

2.5. Claim. Let

$$\begin{aligned} x(3n) &= (x_1, x_2, \dots, x_{3n}) = (1,0,1,\dots,1,0,1) \\ y(3n+1) &= (y_1, \dots, y_{3n+1}) = (1, x_1, \dots, x_{3n}) \text{ and} \\ z(3n+2) &= (z_1, \dots, z_{3n+2}) = (1, 1, x_1, \dots, x_{3n}), \end{aligned} \tag{4}$$

where $x_{3k+1} = x_{3k+3} = 1, x_{3k+2} = 0$, for $k = 0, 1, 2, \dots$. Putting

$$s_j = \sum_{i=1}^{j} 3^{-i} x_i,$$

we have

(i) $\#\langle s_3 \rangle = 2, \#\langle s_6 \rangle = 6,\ \#\langle y(4) \rangle = 3,\ \#\langle y(7) \rangle = 8$ and $\#\langle z(5) \rangle = 4,\ \#\langle z(8) \rangle = 10.$ and

(ii)

$$\#\langle s_{3(n+1)} \rangle = 2\#\langle s_{3n} \rangle + 2\#\langle s_{3(n-1)} \rangle,$$

$$\#\langle y(3(n+1)+1) \rangle = 2\#\langle y(3n+1) \rangle + 2\#\langle y(3(n-1)+1) \rangle$$

$$\text{and } \#\langle z(3(n+1)+2) \rangle = 2\#\langle z(3n+2) \rangle + 2\#\langle z(3(n-1)+2) \rangle,$$

for $n = 1, 2, \dots$.

Proof. (i) Claim (i) is clear.

(ii) We only prove the case $\#\langle s_{3(n+1)}\rangle = 2\#\langle s_{3n}\rangle + 2\#\langle s_{3(n-1)}\rangle$, the other cases are proved similarly. We have

$$\begin{aligned} s_{3n+3} &= s_{3n} + \frac{1}{3^{3n+1}} + \frac{0}{3^{3n+2}} + \frac{1}{3^{3n+1}} \\ &= s_{3n} + \frac{0}{3^{3n+1}} + \frac{1}{3^{3n+2}} + \frac{7}{3^{3n+1}} \\ &= s'_{3n} + \frac{1}{3^{3n+1}} + \frac{7}{3^{3n+2}} + \frac{7}{3^{3n+1}} \\ &= s''_{3n} + \frac{7}{3^{3n+1}} + \frac{7}{3^{3n+2}} + \frac{7}{3^{3n+1}}, \end{aligned} \tag{5}$$

where s_{3n}, s'_{3n} and $s''_{3n} \in$ supp μ_{3n}. Therefore $\#\langle s_{3n+3}\rangle = 2\#\langle s_{3n}\rangle + \#\langle s'_{3n}\rangle + \#\langle s''_{3n}\rangle$. Using Claim 2.2, we have

$$s'_{3n} = s_{3n-3} + \frac{1}{3^{3n-2}} + \frac{0}{3^{3n-1}} + \frac{0}{3^{3n}}, s''_{3n} = s_{3n-3} + \frac{0}{3^{3n+1}} + \frac{0}{3^{3n+2}} + \frac{7}{3^{3n+1}}.$$

So $\#\langle s''_{3n}\rangle = \#\langle s_{3n-3}\rangle$. Assume that $s'_{3n} = s'_{3n-3} + \frac{7}{3^{3n-2}} + \frac{0}{3^{3n-1}} + \frac{0}{3^{3n}}$. Then $x_{3n-3} = 0$, a contradiction to $x_{3n-3} = 1$. Hence $\#\langle s'_{3n}\rangle = \#\langle s_{3n-3}\rangle$. Thus

$$\#\langle s_{3n+3}\rangle = 2\#\langle s_{3n}\rangle + 2\#\langle s_{3n-3}\rangle.$$

The claim is proved.

Putting $F_{3n} = \#\langle s_{3n}\rangle, G_{3n+1} = \#\langle y(3n+1)\rangle$ and $H_{3n+2} = \#\langle z(3n+2)\rangle$, from Claim 2.5 we have

$$\begin{aligned} F_{3n} &= \frac{1}{2\sqrt{3}}[(1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1}], \\ G_{3n+1} &= \frac{1}{4\sqrt{3}}[(1+\sqrt{3})^{n+2} - (1-\sqrt{3})^{n+2}] \text{ and} \\ H_{3n+2} &= \frac{1}{2}[(1+\sqrt{3})^{n+1} + (1-\sqrt{3})^{n+1}]. \end{aligned}$$

2.6. Claim . Let $x = (x_1, x_2, \dots) = (1, 0, 1, \dots, 1, 0, 1, \dots)$ or $x = (1, 1, 0, 1, \dots, 1, 0, 1, \dots)$ or $x = (1, 1, 1, 0, 1, \dots, 1, 0, 1, \dots) \in D^\infty$ and $s = \sum_{i=1}^\infty 3^{-i}x_i \in$ supp μ, we have

$$\alpha(s) = 1 - \frac{\log(1+\sqrt{3})}{3\log 3}.$$

Proof. The proof of the claim is similar to the proof of Claim 2.6. in [10].

We say that $x = (x_1, x_2, \dots, x_n) \in D^n$ is a *maximal sequence* if

$$\#\langle t_n\rangle \leqslant \#\langle s_n\rangle \text{ for any } t_n \in \text{supp } \mu_n,$$

where $s_n = \sum_{i=1}^n 3^{-i}x_i$.

The following fact will be used to estimate the greatest lower bound of local dimension.

2.7. Proposition. *For every* $n \in \mathbb{N}$, *let* $t_{3n+j} = \sum_{i=1}^{3n+j} 3^{-i} t_i$ *be an arbitrary point in supp* μ_{3n+j}, *for* $j = 0, 1, 2$. *Then* $\#\langle t_{3n}\rangle \leqslant F_{3n}, \#\langle t_{3n+1}\rangle \leqslant G_{3n+1}$ *and* $\#\langle t_{3n+2}\rangle \leqslant H_{3n+2}$.

Proof. We will prove the proposition by induction. It is straightforward to check that the assertion holds for $n = 1, 2, 3$. Suppose that it is true for all $n \leqslant k (k \geqslant 3)$. We show that the proposition is true for $n = k+1$. Let $t_{3(k+1)}$ be an arbitrary point in supp μ_{3k+3}. We consider the following cases.

Case 1. $(y_{3k+1}, y_{3k+2}, y_{3k+3})$ is a generator. Without loss of generality, we assume that $(y_{3k+1}, y_{3k+2}, y_{3k+3}) = (1, 0, 1)$. **1.1.** If $y_{3k} = 0$, then $\langle t(3k+3)\rangle = (t(3k-1), 0, 1, 0, 1) \cup (t(3k-1), 0, 0, 1, 7)$.

Hence

$$\#\langle t_{3k+3}\rangle \leqslant 2H_{3(k-1)+2} + G_{3k+1} \leqslant F_{3(k+1)}.$$

1.2. If $y_{3k} = 7$ or $y_{3k} = 1$, then $t(3k+3) = (t(3k), 1, 0, 1) \cup (t(3k), 0, 1, 7)$. It implies

$$\#\langle t_{3k+3}\rangle \leqslant F_{3k} + H_{3k+2} = F_{3(k+1)}.$$

Case 2. $(y_{3k+1}, y_{3k+2}, y_{3k+3})$ is not a generator.

2.1. If $y_{3k+3} = 0$ then by Claim 2.2.(i), inductive hypothesis and (6) we have

$$\#\langle t_{3k+3}\rangle = \#\langle t_{3k+2}\rangle \leqslant H_{3k+2} \leqslant F_{3(k+1)}.$$

2.2.1. Similarity as above, we have if $y_{3k+3} = 1$ and $y_{3k+2} = 1$ or 7, then

$$\#\langle t_{3k+3}\rangle = \#\langle t_{3k+2}\rangle \leqslant H_{3k+2} \leqslant F_{3(k+1)}.$$

2.2.2. If $y_{3k+3} = 1, y_{3k+2} = 0$ and $(y_{3k}, y_{3k+1}, 0, 1)$ is not a generator, then

$$\#\langle t_{3k+3}\rangle \leqslant G_{3k+1} \leqslant F_{3(k+1)}.$$

2.2.3. If $(y_{3k}, y_{3k+1}, 0, 1)$ is a generator, then

$$(y_{3k}, y_{3k+1}, 0, 1) \in \{(0, 7, 0, 1), (1, 0, 0, 1), (1, 7, 0, 1)\}.$$

a) If $(y_{3k}, y_{3k+1}, 0, 1) = (0, 7, 0, 1)$ or $(1, 0, 0, 1)$, then $\#\langle t_{3k+3}\rangle \leqslant 2F_{3k} \leqslant F_{3(k+1)}$.

b) If $(y_{3k}, y_{3k+1}, 0, 1) = (1, 7, 0, 1)$. We consider two cases

b1) If $y_{3k-1} = 7$ or $y_{3k-1} = 1$, then $\#\langle t_{3k+3}\rangle \leqslant 2F_{3k} \leqslant F_{3(k+1)}$.

b2) If $y_{3k-1} = 0$, then $(0, 1, 7, 0, 1) \approx (1, 0, 1, 0, 1)$, hence

$$\langle t(3k+3)\rangle = \langle (y(3k), 1, 0, 1)\rangle$$

for $y(3k) \in D^{3k}$. According to the Case 1, we have

$$\#\langle t(3k+3)\rangle = \#\langle (y(3k), 1, 0, 1)\rangle \leqslant F_{3(k+1)}.$$

2.3. If $y_{3k+3} = 7$. By similar argument as above cases, we get $\#\langle t_{3k+3}\rangle \leqslant F_{3(k+1)}$.

Thus, by using inductive hypothesis and the formula for calculating for F_{3n}, H_{3k+2} and G_{3k+1}, we finished the proof of the first inequality, that means $\#\langle t_{3n}\rangle \leqslant F_{3n}$, for all n.

Similar argument and using the result $\#\langle t_{3n}\rangle \leqslant F_{3n}$, for all n, we have $\#\langle t_{3n+1}\rangle \leqslant G_{3n+1}$ for all n.

By repeating the above argument and using the two above results we have the last of the assertion.

The proposition is proved.

3. Proof of The Main Theorem

We call an infinite sequence $x = (x_1, x_2, \dots) \in D^\infty$ a *prime sequence* if $\#\langle s_n\rangle = 1$ for every n, where $s_n = \sum_{i=1}^{n} 3^{-i} x_i$.

3.1. Claim. $\overline{\alpha} = 1$, $\underline{\alpha} = 1 - \frac{\log(1+\sqrt{3})}{3\log 3}$.

Proof. For any prime sequence $x = (x_1, x_2, \dots)$, for example $x = (x_1, x_2, \dots) = (7, 7, \dots)$ or $x = (x_1, x_2, \dots) = (0, 0, \dots)$, we have $\#\langle s_n\rangle = 1$ for every n, where $s_n = \sum_{i=1}^{n} 3^{-i} x_i$. Therefore, by Proposition 2.4 we get

$$\overline{\alpha} = \overline{\alpha}(s) = \lim_{n\to\infty} \frac{|\log \mu_n(s_n)|}{n\log 3} = 1,$$

where $s = S(x)$.

From Claim 2.6 we have

$$\underline{\alpha} \leqslant 1 - \frac{\log(1+\sqrt{3})}{3\log 3}.$$

For any $t \in \operatorname{supp}\mu$, by Claim 2.5 and Proposition 2.7, we have $\#\langle t_{3n}\rangle \leqslant \#\langle s_{3n}\rangle = \frac{1}{2\sqrt{3}}((1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1})$ for every n. Hence

$$\lim_{n\to\infty} \frac{|\log \mu_{3n}(t_{3n})|}{3n\log 3} \geqslant \lim_{n\to\infty} \frac{|\log \frac{1}{2\sqrt{3}}((1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1})3^{-3n}|}{3n\log 3} = 1 - \frac{\log(1+\sqrt{3})}{3\log 3},$$

where t_n be n - partial sum of t.

Similar argument as above, we have

$$\lim_{n\to\infty} \frac{|\log \mu_{3n+1}(t_{3n+1})|}{(3n+1)\log 3} \geqslant 1 - \frac{\log(1+\sqrt{3})}{3\log 3}, \lim_{n\to\infty} \frac{|\log \mu_{3n+2}(t_{3n+2})|}{(3n+2)\log 3} \geqslant 1 - \frac{\log(1+\sqrt{3})}{3\log 3}.$$

So we get

$$\underline{\alpha} \geqslant 1 - \frac{\log(1+\sqrt{3})}{3\log 3}.$$

Therefore

$$\underline{\alpha} = 1 - \frac{\log(1+\sqrt{3})}{3\log 3}.$$

The claim is proved.

Now we will show that for any $\beta \in (1-\frac{\log(1+\sqrt{3})}{3\log 3}, 1)$ there exists $s \in \operatorname{supp}\mu$ for which $\alpha(s)=\beta$. Let $r = 3(1-\beta)\frac{\log 3}{\log(1+\sqrt{3})}$. Clearly $0<r<1$. For $i = 1, 2, \ldots$, define

$$k_i = \begin{cases} 3i & \text{if } \ i \text{ is odd;} \\ [\frac{3i(1-r)}{r}] & \text{if } \ i \text{ is even,} \end{cases}$$

where $[x]$ denotes the largest integer $\leqslant x$. Let $n_j = \sum_{i=1}^{j} k_i$ and let

$$E_j = \{i : i \leqslant j \text{ and i is even}\} \ ; \ O_j = \{i : i \leqslant j \text{ and i is odd}\},$$
$$e_j = \sum_{i\in E_j} k_i \ ; \ o_j = \sum_{i \in O_j} k_i.$$

Then $n_j = o_j + e_j$.

3.2. Claim. With the above notation we have

$$\lim_{j\to\infty} \frac{j}{n_j} = 0 \ ; \ \lim_{j\to\infty}\frac{n_{j-1}}{n_j} = 1 \text{ and } \lim_{j\to\infty}\frac{o_j}{n_j} = r.$$

Proof. The proof of the claim is similar to the proof of Claim 3.2. in [10].

We define $s \in \operatorname{supp}\mu$ by $s = S(x)$, where

$$x = (\underbrace{1,0,1}_{k_1=3}\underbrace{0,0,\ldots,0}_{k_2}\underbrace{1,0,1,1,0,1,1,0,1}_{k_3=9}\underbrace{0,0,\ldots,0}_{k_4},\ldots). \tag{6}$$

Note that, for $i \in O_j$,

$$\#\langle s_{k_i}\rangle = \frac{1}{2\sqrt{3}}[(1+\sqrt{3})^{\frac{k_i}{3}+1} - (1-\sqrt{3})^{\frac{k_i}{3}+1}]\begin{cases} > \frac{1}{2\sqrt{3}}(1+\sqrt{3})^{\frac{k_i}{3}+1} \\ < \frac{1}{2\sqrt{3}}(1+\sqrt{3})^{\frac{k_i}{3}+2}. \end{cases} \tag{7}$$

Let $s \in \operatorname{supp}\mu$ be defined (6) and let $n_{j-1} \leqslant n < n_j$. By the multiplication principle, we have

$$\prod_{i\in O_{j-1}} \#\langle s_{k_i}\rangle \leqslant \#\langle s_n\rangle \leqslant \prod_{i\in O_j}\#\langle s_{k_i}\rangle.$$

Hence, by (7) yield

$$(\frac{1}{2\sqrt{3}})^{\frac{j-1}{2}}(1+\sqrt{3})^{\frac{o_{j-1}}{3}+\frac{j-1}{2}} \leqslant \#\langle s_n\rangle \leqslant (\frac{1}{2\sqrt{3}})^{\frac{j+1}{2}}(1+\sqrt{3})^{\frac{o_j}{3}+(j+1)},$$

which implies

$$\frac{\log[(\frac{1}{2\sqrt{3}})^{\frac{j-1}{2}}(1+\sqrt{3})^{\frac{o_{j-1}}{3}+\frac{j-1}{2}}]}{n_j\log 3} \leqslant \frac{\log\#\langle s_n\rangle}{n\log 3} \leqslant \frac{\log[(\frac{1}{2\sqrt{3}})^{\frac{j+1}{2}}(1+\sqrt{3})^{\frac{o_j}{3}+(j+1)}]}{n_{j-1}\log 3}.$$

From the latter and Claim 3.1 we get

$$\lim_{n\to\infty} \frac{\log \#\langle s_n\rangle}{n\log 3} = \frac{r}{3}\frac{\log(1+\sqrt{3})}{3\log 3}.$$

Therefore

$$\begin{aligned}\alpha(s) = \lim_{n\to\infty} \frac{|\log \#\langle s_n\rangle 3^{-n}|}{n\log 3} &= 1 - \lim_{n\to\infty} \frac{\log \#\langle s_n\rangle}{n\log 3}\\ &= 1 - \frac{r}{3}\frac{\log(1+\sqrt{3})}{\log 3} = \beta.\end{aligned}$$

The Main Theorem is proved.

References

1. K. J. Falconer, *Techniques in Fractal Geometry,* John Wiley & Sons, 1997.
2. K. J. Falconer, *Fractal Geometry, Mathematical Foundations and Applications,*John Wiley & Sons, 1993.
3. A. Fan, K. S. Lau and S. M. Ngai, Iterated function systems with overlaps, *Asian J. Math.* **4**(2000), 527 - 552.
4. T. Hu, The local dimensions of the Bernoulli convolution associated with the golden number, *Trans. Amer. Math. Soc.* **349**(1997), 2917 - 2940.
5. T. Hu and N. Nguyen, Local dimensions of fractal probability measures associated with equal probability weight, *Preprint.*
6. T. Hu, N. Nguyen and T. Wang, Local dimensions of the probability measure associated with the (0, 1, 3) - problem, *Preprint.*
7. T. Hu, Some open questions related to Probability, Fractal, Wavelets, *East - West J. of Math.* Vol 2, No **1**(2000), 55-71.
8. J. C. Lagarias and Y. Wang, Tiling the line with translates of one tile, *Inventions Math. 124*(1996), 341 - 365.
9. S. M. Ngai and Y. Wang, Hausdorff dimention of the self - similar sets with overlaps,*J. London Math. Soc.* (to appear).
10. Le Xuan Son, Pham Quang Trinh and Vu Hong Thanh. Local Dimension of Fractal Measure associated with the (0,1,a)-Problem, the case $a = 6$.*Journal of Science, VNU*, T.XXI, **1**(2005), 31-44.