# Introduction to LINEAR ALGEBRA FOURTH EDITION



# **GILBERT STRANG**

## INTRODUCTION TO LINEAR ALGEBRA

Fourth Edition

### **GILBERT STRANG**

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### Introduction to Linear Algebra, 4th Edition

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The website for this book is **math.mit.edu/linearalgebra**. A Solutions Manual is available to instructors by email from the publisher.

Course material including syllabus and Teaching Codes and exams and also videotaped lectures are available on the teaching website: web.mit.edu/18.06 Linear Algebra is included in MIT's OpenCourseWare site ocw.mit.edu. This provides video lectures of the full linear algebra course 18.06. MATLAB® is a registered trademark of The MathWorks. Inc.

The front cover captures a central idea of linear algebra. Ax = b is solvable when b is in the (orange) column space of A. One particular solution y is in the (red) row space: Ay = b. Add any vector z from the (green) nullspace of A: Az = 0. The complete solution is x = y + z. Then Ax = Ay + Az = b. The cover design was the inspiration of a creative collaboration: Lois Sellers (birchdesignassociates.com) and Gail Corbett.

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