

# On the martingale representation theorem and approximate hedging a contingent claim in the minimum mean square deviation criterion

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Received 15 November 2006; received in revised form 12 September 2007

**Abstract.** In this work we consider the problem of the approximate hedging of a contingent claim in minimum mean square deviation criterion. A theorem on martingale representation in the case of discrete time and an application of obtained result for semi-continuous market model are given.

*Keywords:* Hedging, contingent claim, risk neutral martingale measure, martingale representation.

## 1. Introduction

The activity of a stock market takes place usually in discrete time. Unfortunately such markets with discrete time are in general incomplete and so super-hedging a contingent claim requires usually an initial price too great, which is not acceptable in practice.

The purpose of this work is to propose a simple method for approximate hedging a contingent claim or an option in minimum mean square deviation criterion.

### *Financial market model with discrete time:*

Without loss of generality let us consider a market model described by a sequence of random vectors  $\{S_n, n = 0, 1, \dots, N\}$ ,  $S_n \in R^d$ , which are discounted stock prices defined on the same probability space  $\{\Omega, \mathfrak{F}, P\}$  with  $\{F_n, n = 0, 1, \dots, N\}$  being a sequence of increasing sigma-algebras of information available up to the time  $n$ , whereas "risk free" asset chosen as a numeraire  $S_n^0 = 1$ .

A  $F_N$ -measurable random variable  $H$  is called a contingent claim (in the case of a standard call option  $H = \max(S_N - K, 0)$ ,  $K$  is strike price).

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**Trading strategy:**

A sequence of random vectors of  $d$ -dimension  $\gamma = (\gamma_n, n = 1, 2, \dots, N)$  with  $\gamma_n = (\gamma_n^1, \gamma_n^2, \dots, \gamma_n^d)^T$  ( $A^T$  denotes the transpose of matrix  $A$ ), where  $\gamma_n^j$  is the number of securities of type  $j$  kept by the investor in the interval  $[n - 1, n)$  and  $\gamma_n$  is  $F_{n-1}$ -measurable (based on the information available up to the time  $n - 1$ ), then  $\{\gamma_n\}$  is said to be predictable and is called *portfolio or trading strategy*.

**Assumptions:**

Suppose that the following conditions are satisfied:

- i)  $\Delta S_n = S_n - S_{n-1}, H \in L_2(P), n = 0, 1, \dots, N.$
- ii) Trading strategy  $\gamma$  is self-financing, i.e.  $S_{n-1}^T \gamma_{n-1} = S_{n-1}^T \gamma_n$  or equivalently  $S_{n-1}^T \Delta \gamma_n = 0$  for all  $n = 1, 2, \dots, N.$

Intuitively, this means that the portfolio is always rearranged in such a way its present value is preserved.

- iii) The market is of *free arbitrage*, that means there is no trading strategy  $\gamma$  such that  $\gamma_1^T S_0 := \gamma_1 \cdot S_0 \leq 0, \gamma_N \cdot S_N \geq 0, P\{\gamma_N \cdot S_N > 0\} > 0.$

This means that with such trading strategy one need not an initial capital, but can get some profit and this occurs usually as the asset  $\{S_n\}$  is not rationally priced.

Let us consider

$$G_N(\gamma) = \sum_{k=1}^N \gamma_k \cdot \Delta S_k \text{ with } \gamma_k \cdot \Delta S_k = \sum_{j=1}^d \gamma_k^j \Delta S_k^j.$$

This quantity is called the *gain* of the strategy  $\gamma$ .

The problem is to find a constant  $c$  and  $\gamma = (\gamma_n, n = 1, 2, \dots, N)$  so that

$$E_P(H - c - G_N(\gamma))^2 \rightarrow \min. \tag{1}$$

Problem (1) have been investigated by several authors such as H.folmer, M.Schweiser, M.Schal, M.L.Nechaev with  $d = 1$ . However, the solution of problem (1) is very complicated and difficult for application if  $\{S_n\}$  is not a  $\{\mathbf{F}_n\}$ -martingale under  $P$ , even for  $d = 1$ .

By the fundamental theorem of financial mathematics, since the market is of free arbitrage, there exists a probability measure  $Q \sim P$  such that under  $Q$   $\{S_n\}$  is an  $\{\mathbf{F}_n\}$ -martingale, i.e.  $E_Q(S_n|F_n) = S_{n-1}$  and the measure  $Q$  is called *risk neutral martingale probability measure*.

We try to find  $c$  and  $\gamma$  so that

$$E_Q(H - c - G_N(\gamma))^2 \rightarrow \min \text{ over } \gamma. \tag{2}$$

**Definition 1.**  $(\gamma_n^*) = (\gamma_n^*(c))$  minimizing the expectation in (1.2) is called  $Q$ - optimal strategy in the minimum mean square deviation (MMSD) criterion corresponding to the initial capital  $c$ .

The solution of this problem is very simple and the construction of the  $Q$ -optimal strategy is easy to implement in practice.

Notice that if  $L_N = dQ/dP$  then

$$E_Q(H - c - G_N(\gamma))^2 = E_P[(H - c - G_N)^2 L_N]$$

can be considered as an weighted expectation under  $P$  of  $(H - c - G_N)^2$  with the weight  $L_N$ . This is similar to the pricing asset based on a risk neutral martingale measure  $Q$ .

In this work we give a solution of the problem (2) and a theorem on martingale representation in the case of discrete time.

It is worth to notice that the authors M.Schweiser, M.Schal, M.L.Nechaev considered only the problem (1) with  $S_n$  of one-dimension and M.Schweiser need the additional assumptions that  $\{S_n\}$  satisfies non-degeneracy condition in the sense that there exists a constant  $\delta$  in  $(0, 1)$  such that

$$(E[\Delta S_n | F_{n-1}])^2 \leq \delta E[(\Delta S_n)^2 | F_{n-1}] \quad \text{P-a.s. for all } n = 1, 2, \dots, N.$$

and the trading strategies  $\gamma_n$ 's satisfy :

$$E[\gamma_n \Delta S_n]^2 < \infty,$$

while in this article  $\{S_n\}$  is of  $d$ -dimension and we need not the preceding assumptions.

The organization of this article is as follows:

The solution of the problem (2) is fulfilled in paragraph 2.(Theorem 1) and a theorem on the representation of a martingale in terms of the differences  $\Delta S_n$  (Theorem 2) will be also given (the representation is similar to the one of a martingale adapted to a Wiener filter in the case of continuous time).

Some examples are given in paragraph 3.

The semi-continuous model, a type of discretization of diffusion model, is investigated in paragraph 4.

## 2. Finding the optimal portfolio

**Notation.** Let  $Q$  be a probability measure such that  $Q$  is equivalent to  $P$  and under  $Q$   $\{S_n, n = 1, 2, \dots, N\}$  is an integrable square martingale and let us denote  $E_n(X) = E_Q(X | F_n)$ ,  $H_N = H$ ,  $H_n = E_Q(H | F_n) = E_n(H)$ ;  $\text{Var}_{n-1}(X) = [\text{Cov}_{n-1}(X_i, X_j)]$  denotes the conditional variance matrix of random vector  $X$  when  $F_{n-1}$  is given,  $\Gamma$  is the family of all predictable strategies  $\gamma$ .

**Theorem 1.** *If  $\{S_n\}$  is an  $\{F_n\}$ -martingale under  $Q$  then*

$$E_Q(H - H_0 - G_N(\gamma^*))^2 = \min\{E_Q(H - c - G_N(\gamma))^2 : \gamma \in \Gamma\}, \tag{3}$$

where  $\gamma_n^*$  is a solution of the following equation system:

$$[\text{Var}_{n-1}(\Delta S_n)]\gamma_n^* = E_{n-1}((\Delta H_n \Delta S_n)) \quad \text{P- a.s.}, \tag{4}$$

*Proof.* At first let us notice that the right side of (3) is finite. In fact, with  $\gamma_n = 1$  for all  $n$ , we have

$$E_Q(H - c - G_N(\gamma))^2 = E_Q \left( H - c - \sum_{n=1}^N \sum_{j=1}^d \Delta S_n^j \right)^2 < \infty.$$

Furthermore, we shall prove that  $\gamma^* \Delta S_n$  is integrable square under  $Q$ .

Recall that (see [Appendix A]) if  $Y, X_1, X_2, \dots, X_d$  are  $d+1$  integrable square random variables with  $E(Y) = E(X_1) = \dots = E(X_d) = 0$  and if  $\hat{Y} = b_1 X_1 + b_2 X_2 + \dots + b_d X_d$  is the optimal linear predictor of  $Y$  on the basis of  $X_1, X_2, \dots, X_d$  then the vector  $b = (b_1, b_2, \dots, b_d)^T$  is the solution of the following equations system :

$$\text{Var}(X)b = E(YX), \tag{5}$$

and as  $\text{Var}(X)$  is non-degenerated  $b$  is defined by

$$b = [\text{Var}(X)]^{-1}E(YX), \tag{6}$$

and in all cases

$$b^T E(YX) \leq E(Y^2), \tag{7}$$

where  $X = (X_1, X_2, \dots, X_k)^T$ .

Furthermore,

$$Y - \widehat{Y} \perp X_i, \text{ i.e. } E[X_i(Y - \widehat{Y})] = 0, \quad i = 1, \dots, k. \tag{8}$$

Applying the above results to the problem of conditional linear prediction of  $\Delta H_n$  on the basis of  $\Delta S_n^1, \Delta S_n^2, \dots, \Delta S_n^d$  as  $F_n$  is given we obtain from (5) the formula (4) defining the regression coefficient vector  $\gamma^*$ . On the other hand we have from (5) and (7):

$$\begin{aligned} E(\gamma_n^{*T} \Delta S_n)^2 &= EE_{n-1}(\gamma_n^{*T} \Delta S_n \Delta S_n^T \gamma_n^{*T}) = E(\gamma_n^{*T} \text{Var}_{n-1}(\Delta S_n) \gamma_n) \\ &= E(\gamma_n^* E_{n-1}(\Delta H_n \Delta S_n)) \leq E(\Delta H_n)^2 < \infty. \end{aligned}$$

With the above remarks we can consider only, with no loss of generality, trading strategies  $\gamma_n$  such that

$$E_{n-1}(\gamma_n \Delta S_n)^2 < \infty.$$

We have:

$$H_N = H_0 + \Delta H_1 + \dots + \Delta H_N$$

and

$$E_{n-1}(\Delta H_n - \gamma_n^T \Delta S_n)^2 = E_{n-1}(\Delta H_n)^2 - 2\gamma_n^T E_{n-1}((\Delta H_n \Delta S_n) + \gamma_n^T E_{n-1}(\Delta S_n \Delta S_n^T) \gamma_n).$$

This expression takes the minimum value when  $\gamma_n = \gamma_n^*$ .

Furthermore, since  $\{H_n - c - G_n(\gamma)\}$  is an  $\{F_n\}$ - integrable square martingale under  $Q$ ,

$$\begin{aligned} E_Q(H_N - c - G_N(\gamma))^2 &= E_Q \left[ H_0 - c - \sum_{n=1}^N (\Delta H_n - \gamma_n \Delta S_n) \right]^2 \\ &= (H_0 - c)^2 + E_Q \left[ \sum_{n=1}^N (\Delta H_n - \gamma_n \Delta S_n) \right]^2 \\ &= (H_0 - c)^2 + \sum_{n=1}^N E_Q(\Delta H_n - \gamma_n \Delta S_n)^2 \text{ (for } \Delta H_n - \gamma_n \Delta S_n \text{ being a martingale difference)} \\ &= (H_0 - c)^2 + E_Q \sum_{n=1}^N E_{n-1}(\Delta H_n - \gamma_n \Delta S_n)^2 \\ &\geq (H_0 - c)^2 + E_Q \sum_{n=1}^N E_{n-1}(\Delta H_n - \gamma_n^* \Delta S_n)^2 \end{aligned}$$

$$\begin{aligned}
 &= (H_0 - c)^2 + E_Q \sum_{n=1}^N (\Delta H_n - \gamma_n^* \Delta S_n)^2 \\
 &= (H_0 - c)^2 + E_Q \left[ \sum_{n=1}^N (\Delta H_n - \gamma_n^* \Delta S_n) \right]^2 \\
 &\geq E_Q (H_N - H_0 - G_n(\gamma^*))^2.
 \end{aligned}$$

So  $E_Q(H_N - c - G_N(\gamma))^2 \geq E_Q(H_N - H_0 - G_n(\gamma^*))^2$  and the inequality becomes the equality if  $c = H_0$  and  $\gamma = \gamma^*$ .

### 3. Martingale representation theorem

**Theorem 2.** Let  $\{H_n, n = 0, 1, 2, \dots\}$ ,  $\{S_n, n = 0, 1, 2, \dots\}$  be arbitrary integrable square random variables defined on the same probability space  $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ ,  $F_n^S = \sigma(S_0, \dots, S_n)$ . Denoting by  $\Pi(S, P)$  the set of probability measures  $Q$  such that  $Q \sim P$  and that  $\{S_n\}$  is  $\{F_n^S\}$  integrable square martingale under  $Q$ , then if  $F = \sqrt{\sum_{n=0}^\infty F_n^S}$ ,  $H_n, S_n \in L_2(Q)$  and if  $\{H_n\}$  is also a martingale under  $Q$  we have:

$$1. H_n = H_0 + \sum_{k=1}^n \gamma_k^T \Delta S_k + C_n \quad a.s., \tag{9}$$

where  $\{C_n\}$  is a  $\{F_n^S\}$ - $Q$ -martingale orthogonal to the martingale  $\{S_n\}$ , i.e.  $E_{n-1}((\Delta C_n \Delta S_n)) = 0$ , for all  $n = 0, 1, 2, \dots$ , whereas  $\{\gamma_n\}$  is  $\{F_{n-1}^S\}$ -predictable.

$$2. H_n = H_0 + \sum_{k=1}^n \gamma_k^T \Delta S_k := H_0 + G_n(\gamma) \quad P\text{-a.s.} \tag{10}$$

for all  $n$  finite iff the set  $\Pi(S, P)$  consists of only one element.

*Proof.* According to the proof of Theorem 1, Putting

$$\Delta C_k = \Delta H_k - \gamma_k^{*T} \Delta S_k, \quad C_n = \sum_{k=1}^n \Delta C_k, \quad C_0 = 0, \tag{11}$$

then  $\Delta C_k \perp \Delta S_k$ , by (8).

Taking summation of (11) we obtain (9).

The conclusion 2 follows from the fundamental theorem of financial mathematics.

**Remark 3.1.** By the fundamental theorem of financial mathematics a security market has no arbitrage opportunity and is complete iff  $\Pi(S, P)$  consists of the only element and in this case we have (10) with  $\gamma$  defined by (4). Furthermore, in this case the conditional probability distribution of  $S_n$  given  $F_{n-1}^S$  concentrates at most  $d + 1$  points of  $R^d$  (see [2], [3]), in particular for  $d = 1$ , with exception of binomial or generalized binomial market models (see [2], [7]), other models are incomplete.

**Remark 3.2.** We can choose the risk neutral martingale probability measure  $Q$  so that  $Q$  has minimum entropy in  $\Pi(S, P)$  as in [2] or  $Q$  is near  $P$  as much as possible.

**Example 1.** Let us consider a stock with the discounted price  $S_0$  at  $t = 0$ ,  $S_1$  at  $t = 1$ , where

$$S_1 = \begin{cases} 4S_0/3 & \text{with prob. } p_1, \\ S_0 & \text{with prob. } p_2, \quad p_1, p_2, p_3 > 0, \quad p_1 + p_2 + p_3 = 1 \\ 5S_0/6 & \text{with prob. } p_3. \end{cases}$$

Suppose that there is an option on the above stock with the maturity at  $t = 1$  and with strike price  $K = S_0$ . We shall show that there are several probability measures  $Q \sim P$  such that  $\{S_0, S_1\}$  is, under  $Q$ , a martingale or equivalently  $E_Q(\Delta S_1) = 0$ .

In fact, suppose that  $Q$  is a probability measure such that under  $Q$   $S_1$  takes the values  $4S_0/3, S_0, 2S_0/3$  with positive probability  $q_1, q_2, q_3$  respectively. Then  $E_Q(\Delta S_1) = 0 \Leftrightarrow S_0(q_1/3 - q_3/6) = 0 \Leftrightarrow 2q_1 = q_3$ , so  $Q$  is defined by  $(q_1, 1 - 3q_1, 2q_1)$ ,  $0 < q_1 < 1/3$ .

In the above market, the payoff of the option is

$$H = (S_1 - K)_+ = (\Delta S_1)_+ = \max(\Delta S_1, 0).$$

It is easy to get an  $Q$ -optimal portfolio

$$\begin{aligned} \gamma^* &= E_Q[H \Delta S_1] / E_Q(\Delta S_1)^2 = 2/3, \quad E_Q(H) = q_1 S_0/3, \\ E_Q[H - E_Q(H) - \gamma^* \Delta S_1]^2 &= q_1 S_0^2 (1 - 3q_1) / 9 \rightarrow 0 \text{ as } q_1 \rightarrow 1/3. \end{aligned}$$

However we can not choose  $q_1 = 1/3$ , because  $q = (1/3, 0, 2/3)$  is not equivalent to  $P$ . It is better to choose  $q_1 \cong 1/3$  and  $0 < q_1 < 1/3$ .

**Example 2.** Let us consider a market with one risky asset defined by :

$$S_n = S_0 \prod_{i=1}^n Z_i, \text{ or } S_n = S_{n-1} Z_n, \quad n = 1, 2, \dots, N,$$

where  $Z_1, Z_2, \dots, Z_N$  are the sequence of i.i.d. random variables taking the values in the set  $\Omega = \{d_1, d_2, \dots, d_M\}$  and  $P(Z_i = d_k) = p_k > 0, k = 1, 2, \dots, M$ . It is obvious that a probability measure  $Q$  is equivalent to  $P$  and under  $Q$   $\{S_n\}$  is a martingale if and only if  $Q\{Z_i = d_k\} = q_k > 0, k = 1, 2, \dots, M$  and  $E_Q(Z_i) = 1$ , i.e.

$$q_1 d_1 + q_2 d_2 + \dots + q_M d_M = 1.$$

Let us recall the integral Hellinger of two measure  $Q$  and  $P$  defined on some measurable space  $\{\Omega^*, F\}$ :

$$H(P, Q) = \int_{\Omega^*} (dP.dQ)^{1/2}.$$

In our case we have

$$\begin{aligned} H(P, Q) &= \sum \{P(Z_1 = d_{i_1}, Z_2 = d_{i_2}, \dots, Z_N = d_{i_N})^* Q(Z_1 = d_{i_1}, Z_2 = d_{i_2}, \dots, Z_N = d_{i_N})\}^{1/2} \\ &= \sum \{p_{i_1} q_{i_1} p_{i_2} q_{i_2} \dots p_{i_N} q_{i_N}\}^{1/2} \end{aligned}$$

where the summation is extended over all  $d_{i_1}, d_{i_2}, \dots, d_{i_N}$  in  $\Omega$  or over all  $i_1, i_2, \dots, i_N$  in  $\{1, 2, \dots, M\}$ . Therefore

$$H(P, Q) = \left\{ \sum_{i=1}^M (p_i q_i)^{1/2} \right\}^N.$$

We can define a distance between  $P$  and  $Q$  by

$$\|Q - P\|^2 = 2(1 - H(P, Q)).$$

Then we want to choose  $Q^*$  in  $\Pi(S, P)$  so that  $\|Q^* - P\| = \inf\{\|Q - P\| : Q \in \Pi(S, P)\}$  by solving the following programming problem:

$$\sum_{i=1}^M p_i^{1/2} q_i^{1/2} \rightarrow \max$$

with the constraints :

- i)  $q_1 d_1 + q_2 d_2 + \dots + q_M d_M = 1.$
- ii)  $q_1 + q_2 + \dots + q_M = 1.$
- iii)  $q_1, q_2, \dots, q_M > 0.$

Giving  $p_1, p_2, \dots, p_M$  we can obtain a numerical solution of the above programming problem. It is possible that the above problem has not a solution. However, we can replace the condition (3) by the condition

$$\text{iii')} \quad q_1, q_2, \dots, q_M \geq 0,$$

then the problem has always the solution  $q^* = (q_1^*, q_2^*, \dots, q_M^*)$  and we can choose the probabilities  $q_1, q_2, \dots, q_M > 0$  are sufficiently near to  $q_1^*, q_2^*, \dots, q_M^*$ .

#### 4. Semi-continuous market model (discrete in time continuous in state)

Let us consider a financial market with two assets:

+ Free risk asset  $\{B_n, n = 0, 1, \dots, N\}$  with dynamics

$$B_n = \exp\left(\sum_{k=1}^n r_k\right), \quad 0 < r_n < 1. \tag{12}$$

+ Risky asset  $\{S_n, n = 0, 1, \dots, N\}$  with dynamics

$$S_n = S_0 \exp\left(\sum_{k=1}^n [\mu(S_{k-1}) + \sigma(S_{k-1})g_k]\right), \tag{13}$$

where  $\{g_n, n = 0, 1, \dots, N\}$  is a sequence of i.i.d. normal random variable  $N(0, 1)$ . It follows from (13) that

$$S_n = S_{n-1} \exp(\mu(S_{n-1}) + \sigma(S_{n-1})g_n), \tag{14}$$

where  $S_0$  is given and  $\mu(S_{n-1}) := a(S_{n-1}) - \sigma^2(S_{n-1})/2$ , with  $a(x), \sigma(x)$  being some functions defined on  $[0, \infty)$ .

The discounted price of risky asset  $\tilde{S}_n = S_n/B_n$  is equal to

$$\tilde{S}_n = S_0 \exp\left(\sum_{k=1}^n [\mu(S_{k-1}) - r_k + \sigma(S_{k-1})g_k]\right). \tag{15}$$

We try to find a martingale measure  $Q$  for this model.

It is easy to see that  $E_P(\exp(\lambda g_k)) = \exp(\lambda^2/2)$ , for  $g_k \sim N(0, 1)$ , hence

$$E \exp\left(\sum_{k=1}^n [\beta_k(S_{k-1})g_k - \beta_k(S_{k-1})^2/2]\right) = 1 \tag{16}$$

for all random variable  $\beta_k(S_{k-1})$ .

Therefore, putting

$$L_n = \exp \left( \sum_{k=1}^n [\beta_k(S_{k-1})g_k - \beta_k(S_{k-1})^2/2] \right), \quad n = 1, \dots, N \tag{17}$$

and if  $Q$  is a measure such that  $dQ = L_N dP$  then  $Q$  is also a probability measure. Furthermore,

$$\frac{\tilde{S}_n}{\tilde{S}_{n-1}} = \exp(\mu(S_{n-1}) - r_n + \sigma(S_{n-1})g_n). \tag{18}$$

Denoting by  $E^0, E$  expectation operations corresponding to  $P, Q$ ,  $E_n(\cdot) = E[(\cdot)|F_n^S]$  and choosing

$$\beta_n = -\frac{(a(S_{n-1}) - r_n)}{\sigma(S_{n-1})} \tag{19}$$

then it is easy to see that

$$E_{n-1}[\tilde{S}_n/\tilde{S}_{n-1}] = E^0[L_n \tilde{S}_n/\tilde{S}_{n-1}|F_n^S]/L_{n-1} = 1$$

which implies that  $\{\tilde{S}_n\}$  is a martingale under  $Q$ .

Furthermore, under  $Q$ ,  $S_n$  can be represented in the form

$$S_n = S_{n-1} \exp((\mu^*(S_{n-1}) + \sigma(S_{n-1})g_n^*). \tag{20}$$

Where  $\mu^*(S_{n-1}) = r_n - \sigma^2(S_{n-1})/2$ ,  $g_n^* = -\beta_n + g_n$  is Gaussian  $N(0, 1)$ . It is not easy to show the structure of  $\Pi(S, P)$  for this model.

We can choose a such probability measure  $E$  or the weight function  $L_N$  to find a  $Q$ - optimal portfolio.

**Remark 4.3.** The model (12), (13) is a type of discretization of the following diffusion model:

Let us consider a financial market with continuous time consisting of two assets:

+Free risk asset:

$$B_t = \exp \left( \int_0^t r(u)du \right). \tag{21}$$

+Risky asset:  $dS_t = S_t[a(S_t)dt + \sigma(S_t)dW_t]$ ,  $S_0$  is given, where  $a(\cdot), \sigma(\cdot) : (0, \infty) \rightarrow R$  such that  $xa(x), x\sigma(x)$  are Lipschitz. It is obvious that

$$S_t = \exp \left\{ \int_0^t [a(S_u) - \sigma^2(S_u)/2]du + \int_0^t \sigma(S_u)dW_u \right\}, \quad 0 \leq t \leq T. \tag{22}$$

Putting

$$\mu(S) = a(S) - \sigma^2(S)/2, \tag{23}$$

and dividing  $[0, T]$  into  $N$  intervals by the equidistant dividing points  $0, \Delta, 2\Delta, \dots, N\Delta$  with  $N = T/\Delta$  sufficiently great, it follows from (21), (22) that

$$\begin{aligned} S_{n\Delta} &= S_{(n-1)\Delta} \exp \left\{ \int_{(n-1)\Delta}^{n\Delta} \mu(S_u)du + \int_{(n-1)\Delta}^{n\Delta} \sigma(S_u)dW_u \right\} \\ &\cong S_{(n-1)\Delta} \exp\{\mu(S_{(n-1)\Delta})\Delta + (S_{(n-1)\Delta})[W_{n\Delta} - W_{(n-1)\Delta}]\} \\ &\cong S_{(n-1)\Delta} \exp\{\mu(S_{(n-1)\Delta})\Delta + \sigma(S_{(n-1)\Delta})\Delta^{1/2}g_n\} \end{aligned}$$

with  $g_n = [W_{n\Delta} - W_{(n-1)\Delta}]/\Delta^{1/2}$ ,  $n = 1, \dots, N$ , being a sequence of the i.i.d. normal random variables of the law  $N(0, 1)$ , so we obtain the model :

$$S_{n\Delta}^* = S_{(n-1)\Delta}^* \exp\{\mu(S_{(n-1)\Delta}^*)\Delta + \sigma(S_{(n-1)\Delta}^*)\Delta^{1/2}g_n\}. \tag{24}$$

Similarly we have

$$B_{n\Delta}^* \cong B_{(n-1)\Delta}^* \exp(r_{(n-1)\Delta}\Delta). \tag{25}$$

According to (21), the discounted price of the stock  $S_t$  is

$$\tilde{S}_t = \frac{S_t}{B_t} = S_0 \exp\left\{\int_0^t [\mu(S_u) - r_u]du + \int_0^t \sigma(S_u)dW_u\right\}. \tag{26}$$

By Theorem Girsanov, the unique probability measure  $Q$  under which  $\{\tilde{S}_t, F_t^S, Q\}$  is a martingale is defined by

$$(dQ/dP)|_{F_T^S} = \exp\left(\int_0^T \beta_u dW_u - \frac{1}{2}\int_0^T \beta_u^2 du\right) := L_T(\omega), \tag{27}$$

where

$$\beta_s = -\frac{(a(S_s) - r_s)}{\sigma(S_s)},$$

and  $(dQ/dP)|_{F_T^S}$  denotes the Radon-Nikodym derivative of  $Q$  w.r.t.  $P$  limited on  $F_T^S$ . Furthermore, under  $Q$

$$W_t^* = W_t + \int_0^t \beta_u du$$

is a Wiener process. It is obvious that LT can be approximated by

$$L_N := \exp\left(\sum_{k=1}^N \beta_k \Delta^{1/2} g_k - \Delta \beta_k^2 / 2\right) \tag{28}$$

where

$$\beta_n = -\frac{[a(S_{(n-1)\Delta}) - r_{n\Delta}]}{\sigma(S_{(n-1)\Delta})} \tag{29}$$

Therefore the weight function (25) is approximate to Radon-Nikodym derivative of the risk unique neutral martingale measure  $Q$  w.r.t.  $P$  and  $Q$  is used to price derivatives of the market.

Remark 4.4. In the market model Black- Scholes we have  $L_N = L_T$ . We want to show now that for the weight function (28)

$$E_Q(H - H_0 - G_N(\gamma^*))^2 \rightarrow 0 \text{ as } N \rightarrow \infty \text{ or } \Delta \rightarrow 0.$$

where  $\gamma^*$  is  $Q$ -optimal trading strategy.

**Proposition.** Suppose that  $H = H(S_T)$  is a integrable square discounted contingent claim. Then

$$E_Q(H - H_0 - G_N(\gamma^*))^2 \rightarrow 0 \text{ as } N \rightarrow \infty \text{ or } \Delta \rightarrow 0, \tag{30}$$

provided  $a$ ,  $r$  and  $\sigma$  are constant ( in this case the model (21), (22) is the model Black-Scholes ).

*Proof.* It is well known (see[4], [5]) that for the model of complete market (21), (22) there exists a trading strategy  $\varphi = (\varphi_t = \varphi(t, S(t)), 0 \leq t \leq T)$ , hedging  $H$ , where  $\varphi : [0, T] \times (0, \infty) \rightarrow R$  is continuously derivable in  $t$  and  $S$ , such that

$$H(S_T) = H_0 + \int_0^T \varphi_t d\tilde{S}(t) \quad \text{a.s.}$$

On the other hand we have

$$\begin{aligned}
 & E_{Q_N} \left( H - H_0 - \sum_{k=1}^N \gamma_{(k-1)\Delta}^* \Delta \tilde{S}_{n\Delta} \right)^2 \\
 & \leq E_{Q_N} \left( H - H_0 - \sum_{k=1}^N \varphi_{(k-1)\Delta} \Delta \tilde{S}_{n\Delta} \right)^2 \\
 & = E_Q \left( \int_0^T \varphi_t d\tilde{S}(t) - \sum_{k=1}^N \varphi_{(n-1)\Delta} \Delta \tilde{S}_{(n-1)\Delta} \right)^2 L_N/L_T \\
 & = E_Q \left( \int_0^T \varphi_t d\tilde{S}(t) - \sum_{k=1}^N \phi_{(k-1)\Delta} \Delta \tilde{S}_{(n-1)\Delta} \right)^2 \rightarrow 0 \text{ as } \Delta \rightarrow 0.
 \end{aligned}$$

(since  $L_N = L_T$  and by the definition of the stochastic integral Ito as  $a$  and  $\sigma$  are constant ).

**Appendix A**

Let  $Y, X_1, X_2, \dots, X_d$  be integrable square random variables defined on the same probability space  $\{\Omega, F, P\}$  such that  $EX_1 = \dots = EX_d = EY = 0$ .

We try to find a coefficient vector  $b = (b_1, \dots, b_d)^T$  so that

$$E(Y - b_1X_1 - \dots - b_dX_d)^2 = E(Y - b^T X)^2 = \min_{a \in R^d} (Y - a^T X)^2. \tag{A1}$$

Let us denote  $EX = (EX_1, \dots, EX_d)^T$ ,  $\text{Var}(X) = [\text{Cov}(X_i, X_j), i, j = 1, 2, \dots, d] = EXX^T$ .

**Proposition.** The vector  $b$  minimizing  $E(Y - a^T X)^2$  is a solution of the following equation system :

$$\text{Var}(X)b = E(XY). \tag{A2}$$

Putting  $U = Y - b^T X = Y - \hat{Y}$ , with  $\hat{Y} = b^T X$ , then

$$E(U^2) = EY^2 - b^T E(XY) \geq 0. \tag{A3}$$

$$E(UX_i) = 0 \text{ for all } i = 1, \dots, d. \tag{A4}$$

$$EY^2 = EU^2 + E\hat{Y}^2. \tag{A5}$$

$$\rho = \frac{EY\hat{Y}}{(EY^2 E\hat{Y}^2)^{1/2}} = \left( \frac{E\hat{Y}^2}{EY^2} \right)^{1/2}. \tag{A6}$$

( $\rho$  is called multiple correlation coefficient of  $Y$  relative to  $X$ ).

*Proof.* Suppose at first that  $\text{Var}(X)$  is a positively definite matrix. For each  $a \in R^d$  We have

$$F(a) = E(Y - a^T X)^2 = EY^2 - 2a^T E(XY) + a^T EXX^T a \tag{A7}$$

$$\nabla F(a) = -2E(XY) + 2\text{Var}(X)a.$$

$$\left[ \frac{\partial F(a)}{\partial a_i \partial a_j}, i, j = 1, 2, \dots, d \right] = 2\text{Var}(X).$$

It is obvious that the vector  $b$  minimizing  $F(a)$  is the unique solution of the following equation:

$$\nabla F(a) = 0 \text{ or (A2)}$$

and in this case (A2) has the unique solution :

$$b = [\text{Var}(X)]^{-1}E(XY).$$

We assume now that  $1 \leq \text{Rank}(\text{Var}(X)) = r < d$ .

We denote by  $e_1, e_2, \dots, e_d$  the ortho-normal eigenvectors w.r.t. the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_d$  of  $\text{Var}(X)$ , where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0 = \lambda_{r+1} = \dots = \lambda_d$  and  $P$  is a orthogonal matrix with the columns being the eigenvectors  $e_1, e_2, \dots, e_d$ , then we obtain :

$$\text{Var}(X) = P\Lambda P^T, \text{ with } \Lambda = \text{Diag}(\lambda_1, \lambda_2, \dots, \lambda_d).$$

Putting

$$Z = P^T X = [e_1^T X, e_2^T X, \dots, e_d^T X]^T,$$

$Z$  is the principle component vector of  $X$ , we have

$$\text{Var}(Z) = P^T \text{Var}(X) P = \Lambda = \text{Diag}(\lambda_1, \lambda_2, \dots, \lambda_r, 0, \dots, 0).$$

Therefore

$$EZ_{r+1}^2 = \dots = EZ_d^2 = 0, \text{ so } Z_{r+1} = \dots = Z_d = 0 \text{ P- a.s.}$$

Then

$$\begin{aligned} F(a) &= E(Y - a^T X)^2 = E(Y - (a^T P) Z)^2 \\ &= E(Y - a_1^* Z_1 - \dots - a_d^* Z_d)^2 \\ &= E(Y - a_1^* Z_1 - \dots - a_r^* Z_r)^2. \end{aligned}$$

where

$$a^{*T} = (a_1^*, \dots, a_d^*) = a^T P, \text{ Var}(Z_1, \dots, Z_r) = \text{Diag}(\lambda_1, \lambda_2, \dots, \lambda_r) > 0.$$

According to the above result  $(b_1^*, \dots, b_r^*)^T$  minimizing  $E(Y - a_1^* Z_1 - \dots - a_r^* Z_r)^2$  is the solution of

$$\begin{pmatrix} \lambda_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \lambda_r \end{pmatrix} \begin{pmatrix} b_1^* \\ \dots \\ b_r^* \end{pmatrix} = \begin{pmatrix} EZ_1 Y \\ \dots \\ EZ_r Y \end{pmatrix} \tag{A8}$$

or

$$\begin{pmatrix} \lambda_1 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \lambda_r & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} b_1^* \\ \dots \\ b_r^* \\ b_{r+1}^* \\ \dots \\ b_d^* \end{pmatrix} = \begin{pmatrix} EZ_1 Y \\ \dots \\ EZ_r Y \\ 0 \\ \dots \\ 0 \end{pmatrix} = \begin{pmatrix} EZ_1 Y \\ \dots \\ EZ_r Y \\ EZ_{r+1} Y \\ \dots \\ EZ_d Y \end{pmatrix} \tag{A9}$$

with  $b_{r+1}^*, \dots, b_d^*$  arbitrary .

Let  $b = (b_1, \dots, b_d)^T$  be the solution of  $b^T P = b^{*T}$ , hence  $b = P b^*$  with  $b^*$  being a solution of (A9).

Then it is follows from (A9) that

$$\text{Var}(Z) P^T b = E(ZY) = P^T E(XY)$$

or

$$P^T \text{Var}(X) P P^T b = P^T E(XY) \text{ ( since } \text{Var}(Z) = P^T \text{Var}(X) P \text{)}$$

or

$$\text{Var}(X) b = E(XY)$$

which is (A2). Thus we have proved that (A2) has always a solution, which solves the problem (A1). By (A7), we have

$$\begin{aligned} F(b) &= \min_a E(Y - a^T X)^2 \\ &= EY^2 - 2b^T E(XY) + b^T \text{Var}(X)b \\ &= EY^2 - 2b^T E(XY) + b^T E(XY) \\ &= EY^2 - b^T E(XY) \geq 0. \end{aligned}$$

On the other hand

$$EU X_i = E(X_i Y) - E(X_i b^T X) = 0, \quad (\text{A10})$$

since  $b$  is a solution of (A2) and (A10) is the  $i$ th equation of the system (A2).

It follows from (A10) that

$$E(U\hat{Y}) = 0 \text{ and } EY^2 = E(U + \hat{Y})^2 = EU^2 + E\hat{Y}^2 + 2E(U\hat{Y}) = EU^2 + E\hat{Y}^2.$$

**Remark.** We can use Hilbert space method to prove the above proposition. In fact, let  $H$  be the set of all random variables  $\xi$ 's such that  $E\xi = 0$ ,  $E\xi^2 < \infty$ , then  $H$  becomes a Hilbert space with the scalar product  $(\xi, \zeta) = E\xi\zeta$ , and with the norm  $\|\xi\| = (E\xi^2)^{1/2}$ . Suppose that  $X_1, X_2, \dots, X_d, Y \in H$ ,  $L$  is the linear manifold generated by  $X_1, X_2, \dots, X_d$ . We want to find a  $\hat{Y} \in L$  so that  $\|Y - \hat{Y}\|$  minimizes, that means  $\hat{Y} = b^T X$  solves the problem (A1). It is obvious that  $\hat{Y}$  is defined by

$$\hat{Y} = \text{Proj}_L Y = b^T X \text{ and } U = \hat{Y} - Y \in L^\perp.$$

Therefore  $(Y - b^T X, X_i) = 0$  or  $E(b^T X X_i) = E(X_i Y)$  for all  $i = 1, \dots, d$  or  $b^T E(X^T X) = E(XY)$  which is the equation (A2). The rest of the above proposition is proved similarly.

**Acknowledgements.** This paper is based on the talk given at the Conference on Mathematics, Mechanics and Informatics, Hanoi, 7/10/2006, on the occasion of 50th Anniversary of Department of Mathematics, Mechanics and Informatics, Vietnam National University, Hanoi.

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