VNU Journal of Science, Mathematics - Physics 23 (2007) 155-158

The parameter-dependent cyclic inequality

Nguyen Vu Luong*

Department of Mathematics, Mechanics, Informatics, College of Science, VNU
334 Nguyen Trai, Hanoi, Vietnam

Received 15 November 2006; received in revised form 12 September 2007

Abstract. In this paper we will construct a parameter-dependent cyclic inequality that can be used to prove a lot of hard and interesting inequalities.

1. Introduction

The cyclic inequality is a type of inequality that may be right in just some particular cases but not in genenal. In this paper, we propose one type of parameter-dependent cyclic inequality from a special inequality. Thanks to this inequality, we can obtain many inequalities by choosing α and n. Note that it can be proved by some ways in particular case. However in order to prove it in general case, we have to use the method that is mentioned in the paper.

2. The general case

Denote $R^+ = \{x \in Rx > 0\}$.

Lemma 1.1. *Assume that* $x_i \in R, \quad (i = \overline{1,n})$ *we have*

$$\sum_{1 \le i < j \le n} x_i x_j \le \frac{n-1}{2n}\Big(\sum_{i=1}^{n} x_i\Big)^2.$$

Proof. We have

$$\sum_{1 \le i < j \le n} x_i x_j \le \sum_{1 \le i < j \le n} \frac{x_i^2 + x_j^2}{2}.$$

Since $1 + 2 + \cdots + (n-1) = \dfrac{n(n-1)}{2}$, hence the number of terms of $\sum_{1 \le i < j \le n} x_i x_j$ is $\dfrac{n(n-1)}{2}$. It follows

$$2 \sum_{1 \le i < j \le n} x_i x_j \le \sum_{1 \le i < j \le n} (x_i^2 + x_j^2) = (n-1)(\sum_{i=1}^{n} x_i^2).$$

* E-mail: luongnv@vnu.edu.vn

Adding both sides of the above inequality by $2(n-1)\sum_{1\leq i<j\leq n} x_i x_j$, we obtain the inequality as was to be proved.

The proof of Lemma 1.1 is complete.

Theorem 1.1. *Assume that* $x_i \quad (i=\overline{1,n}),\quad n\geqslant 3$ *are positive number. Then there holds the following inequality*

$$\frac{x_1}{x_1+\alpha(x_2+\cdots+c_nx_{k+1})}+\frac{x_2}{x_2+\alpha(x_3+\cdots+c_nx_{k+2})}+\cdots+$$
$$+\cdots+\frac{x_n}{x_n+\alpha(x_1+\cdots+c_nx_k)}\geqslant\frac{2n}{2+\alpha(n-1)}\quad(1.1)$$

Where $c_n=\dfrac{(n\mod 2+1)}{2},\quad k=[\dfrac{n}{2}]$ *and* α *is an arbitrary real number satisfies* $\alpha\geqslant 2$.

Proof. First, for the sake convinience, we set

$$P=\frac{x_1}{x_1+\alpha(x_2+\cdots+c_nx_{k+1})}+\frac{x_2}{x_2+\alpha(x_3+\cdots+c_nx_{k+2})}+\cdots+$$
$$+\cdots+\frac{x_n}{x_n+\alpha(x_1+\cdots+c_nx_k)}\geqslant\frac{2n}{2+\alpha(n-1)}.$$

Now let's consider the case $n=2k+1$ it gives

$$P=\frac{x_1^2}{x_1^2+\alpha(x_1x_2+\cdots+x_1x_{k+1})}+\frac{x_2^2}{x_2^2+\alpha(x_2x_3+\cdots+x_2x_{k+2})}+$$
$$+\cdots+\frac{x_n^2}{x_n^2+\alpha(x_nx_1+\cdots+x_nx_k)}.$$

Using the fact that

$$\sum_{i=1}^{n}\frac{x_i^2}{a_i}\geqslant\frac{(\sum_{i=1}^n x_i)^2}{\sum_{i=1}^n a_i}\quad(1.2)$$

with $a_i\in R^+\quad(i=\overline{1,n})$, it implies

$$P\geqslant\frac{(\sum_{i=1}^n x_i)^2}{\sum_{i=1}^n x_i^2+\alpha\sum_{1\leq i<j\leq n}x_ix_j}.$$

Since $\alpha\geqslant 2$,it can be rewritten as $\alpha=2+\beta$ with $\beta\geqslant 0$. This leads to

$$P\geqslant\frac{(\sum_{i=1}^n x_i)^2}{(\sum_{i=1}^n x_i)^2+\beta\sum_{1\leq i<j\leq n}x_ix_j}.$$

Applying Lemma (1.1) we obtain

$$P\geqslant\frac{(\sum_{i=1}^n x_i)^2}{(\sum_{i=1}^n x_i)^2+\dfrac{\beta(n-1)}{2n}(\sum_{i=1}^n x_i)^2}$$

or $\quad P\geqslant\dfrac{2n}{2+\alpha(n-1)}.$

Next, for $n = 2k$, we get

$$P = \frac{x_1^2}{x_1^2 + \alpha(x_1x_2 + \cdots + x_1x_k + \frac{1}{2}x_1x_{k+1})} + \frac{x_2^2}{x_2^2 + \alpha(x_2x_3 + \cdots + x_2x_{k+1} + \frac{1}{2}x_2x_{k+2})} +$$

$$+ \cdots + \frac{x_n^2}{x_n^2 + \alpha(x_nx_1 + \cdots + x_nx_{k-1} + \frac{1}{2}x_nx_k)}.$$

Applying the inequality (1.2), we get

$$P \geqslant \frac{(\sum_{i=1}^n x_i)^2}{\sum_{i=1} x_i^2 + \alpha \sum_{1 \le i < j \le n} x_ix_j} = \frac{(\sum_{i=1}^n x_i)^2}{(\sum_{i=1} x_i)^2 + \beta \sum_{1 \le i < j \le n} x_ix_j}$$

Using the Lemma 1.1 once more, we come to the following inequality

$$P \geqslant \frac{(\sum_{i=1}^n x_i)^2}{(\sum_{i=1}^n x_i)^2 + \frac{\beta(n-1)}{2n}(\sum_{i=1}^n x_i)^2} = \frac{2n}{2 + \alpha(n-1)}.$$

Thus Theorem 1.1 is proved.

3. The special cases

For $n = 3$, we obtain the following inequalities.

Example 1.1. Let a, b, c be positive numbers, $\alpha \geqslant 2$, prove that

$$\frac{a}{a + \alpha b} + \frac{b}{b + \alpha c} + \frac{c}{c + \alpha a} \geqslant \frac{3}{1 + \alpha}.$$

Take $\alpha = 2$ we obtain

Example 1.2. Let a, b, c be positive numbers, prove that

$$\frac{a}{a + 2b} + \frac{b}{b + 2c} + \frac{c}{c + 2a} \geqslant 1.$$

Take $\alpha = \frac{1}{abc} \geqslant 2 \Leftrightarrow abc \le \frac{1}{2}$, we yield

Example 1.3. Let a, b, c be positive numbers satisfy $abc \le \frac{1}{2}$, prove that

$$\frac{a^2c}{1 + a^2c} + \frac{b^2a}{1 + b^2a} + \frac{c^2b}{1 + c^2b} \geqslant \frac{3abc}{1 + abc}.$$

For $n = 4$ we yield the inequality

Example 1.4. Assume that $a, b, c, d \in R^+, \alpha \geqslant 2$, prove that

$$\frac{a}{2a + \alpha(2b + c)} + \frac{b}{2b + \alpha(2c + d)} + \frac{c}{2c + \alpha(2d + a)} + \frac{d}{2d + \alpha(2a + b)} \geqslant \frac{4}{2 + 3\alpha}.$$

Take $\alpha = 2$ we obtain

Example 1.5. Assume that $a, b, c, d \in R^+$, prove that

$$\frac{a}{2a + 4b + 2c} + \frac{b}{2b + 4c + 2d} + \frac{c}{2c + 4d + 2a} + \frac{d}{2d + 4a + 2b} \geqslant \frac{1}{2}$$

Take $a = b, b = c$ we get

Example 1.5. Assume that $a, b \in R^+, \alpha \geqslant 2$, prove that

$$\frac{a((\alpha+2)a+2\alpha b)}{[2a+\alpha(2a+b)][2a+3\alpha b]} + \frac{b((\alpha+2)b+2\alpha a)}{[2b+\alpha(2b+a)][2b+3\alpha a]} \geqslant \frac{2}{2+3\alpha}.$$

For $n = 5$ we yield the inequality

Example 1.7. Give $a, b, c, d, e \in R^+, \alpha \geqslant 2$, prove that

$$P = \frac{a}{a+\alpha(b+c)} + \frac{b}{b+\alpha(c+d)} + \frac{c}{c+\alpha(d+e)} + \frac{d}{d+\alpha(e+a)} + \frac{e}{e+\alpha(a+b)} \geqslant \frac{5}{1+2\alpha}.$$

Take $c = d = e, \alpha = 2$ we yield the inequality

Example 1.8. Given $a, b, c \in R^+$, prove that

$$\frac{a}{a+2b+2c} + \frac{b}{b+4c} + 2c\Big(\frac{2a+2c+b}{[c+2(c+a)][c+2(a+b)]}\Big) \geqslant \frac{4}{5}.$$

For $n = 6$ we yield

Example 1.9. Given $a_i \in R^+ \quad (i = \overline{1,6}), \alpha \geqslant 2$, prove that

$$\frac{a_1}{a_1+\alpha(a_2+a_3+\frac{1}{2}a_4)} + \frac{a_2}{a_2+\alpha(a_3+a_4+\frac{1}{2}a_5)} + \frac{a_3}{a_3+\alpha(a_4+a_5+\frac{1}{2}a_6)} +$$

$$+ \frac{a_4}{a_4+\alpha(a_5+a_6+\frac{1}{2}a_1)} + \frac{a_5}{a_5+\alpha(a_6+a_1+\frac{1}{2}a_2)} + \frac{a_6}{a_6+\alpha(a_1+a_2+\frac{1}{2}a_3)} \geqslant \frac{12}{2+5\alpha}$$

Finally, take $a_1 = a_2 = a, a_3 = a_4 = b, a_5 = a_6 = c$ and $\alpha = 2$ we get

Example 1.10. Assume that $a, b, c \in R^+$, prove that

$$a\Big(\frac{1}{3a+3b} + \frac{1}{a+4b+c}\Big) + b\Big(\frac{1}{3b+3c} + \frac{1}{b+4c+a}\Big) + c\Big(\frac{1}{3c+3a} + \frac{1}{c+4a+b}\Big) \geqslant 1.$$

Acknowledgements. This paper is based on the talk given at the Conference on Mathematics, Mechanics, and Informatics, Hanoi, 7/10/2006, on the occasion of 50th Anniversary of Department of Mathematics, Mechanics and Informatics, Vietnam National University, Hanoi.

References

[1] B.A Troesch, The cyclic inequality for a large number of terms, *Notices Amer. Math Soc.* 25 (1978).

[2] E.S Freidkin, S.A Freidkin, On a problem by Shapiro, *Elem. Math.* 45 (1990) 137.