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Some results on (IEZ)-modules

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Abstract. A module M is called $(IEZ)-$module if for the submodules A, B, C of M such that $A \cap B = A \cap C = B \cap C = 0$, then $A \cap (B \oplus C) = 0$. It is shown that:

(1) Let $M_1, ..., M_n$ be uniform local modules such that M_i does not embed in $J(M_j)$ for any $i, j = 1, ..., n$. Suppose that $M = M_1 \oplus ... \oplus M_n$ is a $(IEZ)-$module. Then

(a) M satisfies (C_3).

(b) The following assertions are equivalent:

(i) M satisfies (C_2).

(ii) If $X \subseteq M, X \cong M_i$ (with $i \in \{1, ..., n\}$), then $X \subseteq^{\oplus} M$.

(2) Let $M_1, ..., M_n$ be uniform local modules such that M_i does not embed in $J(M_j)$ for any $i, j = 1, ..., n$. Suppose that $M = M_1 \oplus ... \oplus M_n$ is a nonsingular $(IEZ)-$module. Then, M is a continuous module.

1. Introduction

Throughout this note, all rings are associative with identity, and all modules are unital right modules. The Jacobson radical and the endmorphism ring of M are denoted by $J(M)$ and End(M). The notation $X \subseteq^e Y$ means that X is an essential submodule of Y.

For a module M consider the following conditions:

(C_1) Every submodule of M is essential in a direct summand of M.

(C_2) Every submodule isomorphic to a direct summand of M is itself a direct summand.

(C_3) If A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M.

A module M is defined to be a CS-module (or an extending module) if M satisfies the condition (C_1). If M satisfies (C_1) and (C_2), then M is said to be a continuous module. M is called quasi-continuous if it satisfies (C_1) and (C_3). A module M is said to be a uniform - extending if every uniform submodule of M is essential in a direct summand of M. We have the following implications:

We refer to [1] and [2] for background on CS and (quasi-)continuous modules.

In this paper, we give some results on $(IEZ)-$modules with conditions $(C_1), (C_2), (C_3)$.

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2. The results

A module M is called $(IEZ)-$module if for the submodules A, B, C of M such that $A \cap B = A \cap C = B \cap C = 0$, then $A \cap (B \oplus C) = 0$.

Examples

(a) Let F be a field. We consider the ring

$$R = \begin{pmatrix} F & 0 & \dots & 0 \\ 0 & F & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & F \end{pmatrix}$$

Then R_R is a $(IEZ)-$module.

Proof. Let A, B, C be submodules of $M = R_R$ such that $A \cap B = A \cap C = B \cap C = 0$. Then, there exist the subsets I, J, K of $\{1, ..., n\}$ with $I \cap J = I \cap K = J \cap K = \emptyset$ such that

$$A = \begin{pmatrix} A_{11} & 0 & \dots & 0 \\ 0 & A_{22} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & A_{nn} \end{pmatrix}$$

where $A_{ii} = F \ \forall i \in I$, and $A_{ii} = 0 \ \forall i \in I'$, with $I' = \{1, ..., n\} \backslash I$,

$$B = \begin{pmatrix} B_{11} & 0 & \dots & 0 \\ 0 & B_{22} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & B_{nn} \end{pmatrix}$$

where $B_{ii} = F \ \forall i \in J$, and $B_{ii} = 0 \ \forall i \in J'$, with $J' = \{1, ..., n\} \backslash J$,

$$C = \begin{pmatrix} C_{11} & 0 & \dots & 0 \\ 0 & C_{22} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & C_{nn} \end{pmatrix}$$

where $C_{ii} = F \ \forall i \in K$, and $C_{ii} = 0 \ \forall i \in K'$, with $K' = \{1, ..., n\} \backslash K$.

Therefore,

$$B \oplus C = \begin{pmatrix} X_{11} & 0 & \dots & 0 \\ 0 & X_{22} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & X_{nn} \end{pmatrix}$$

where $X_{ii} = F \ \forall i \in (J \cup K)$, and $X_{ii} = 0 \ \forall i \in H$, with $H = \{1, ..., n\} \backslash (J \cup K)$. Since $I \cap (J \cup K) = \emptyset$, thus $A \cap (B \oplus C) = 0$.

Hence R_R is a $(IEZ)-$module.

Remark. Let

$$M_1 = \begin{pmatrix} F & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

....

$$M_n = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & F \end{pmatrix},$$

then M_i which are simple modules for any $i = 1, ..., n$ and $R_R = M_1 \oplus ... \oplus M_n$ where R_R in example. Therefore, M_i are uniform local modules such that M_i does not embed in $J(M_j)$ for any $i, j = 1, ..., n$.

(b) Let F be a field and V is a vector space over field F. Set $M = V \oplus V$. Then M is not $(IEZ)-$module.

Proof. Let $A = \{(x, x) \mid x \in V\}$, $B = V \oplus 0$, $C = 0 \oplus V$ be submodules of M. We have $A \cap B = A \cap C = B \cap C = 0$ but $A \cap (B \oplus C) = A \cap M = A$. Hence, M is not $(IEZ)-$module.

We give two results on $(IEZ)-$module with conditions $(C_1), (C_2), (C_3)$.

Theorem 1. *Let $M_1, ..., M_n$ be uniform local modules such that M_i does not embed in $J(M_j)$ for any $i, j = 1, ..., n$. Suppose that $M = M_1 \oplus ... \oplus M_n$ is $(IEZ)-$module. Then*

(a) M satisfies (C_3).

(b) The following assertions are equivalent:

(i) M satisfies (C_2).

(ii) If $X \subseteq M, X \cong M_i$ (with $i \in \{1, ..., n\}$), then $X \subseteq^{\oplus} M$.

Theorem 2. *Let $M_1, ..., M_n$ be uniform local modules such that M_i does not embed in $J(M_j)$ for any $i, j = 1, ..., n$. Suppose that $M = M_1 \oplus ... \oplus M_n$ is a nonsingular $(IEZ)-$module. Then M is a continuous module.*

3. Proof of Theorem 1 and Theorem 2

Lemma 1. ([3, Lemma1.1]) *Let N be a uniform local module such that N does not embed in $J(N)$, then $S = End(N)$ is a local ring.*

Lemma 2. *Let $M_1, ..., M_n$ be uniform local modules such that M_i does not embed in $J(M_j)$ for any $i, j = 1, ..., n$. Set $M = M_1 \oplus ... \oplus M_n$. If $S_1, S_2 \subseteq^{\oplus} M$; $u-dim(S_1) = 1$ and $u-dim(S_2) = n-1$, then $M = S_1 \oplus S_2$.*

Proof. By Lemma 1 we have $End(M_i)$ which is a local ring for any $i = 1, ..., n$. By Azumaya's Lemma (cf. [4, 12.6, 12.7]), we have $M = S_2 \oplus K = S_2 \oplus M_i$. Suppose that $i = 1$, i.e., $M = S_2 \oplus M_1 = (\oplus_{i=2}^{n} M_i) \oplus M_1$; $M = S_1 \oplus H = S_1 \oplus (\oplus_{i \in I} M_i)$ with $\mid I \mid = n-1$. There are cases:

Case 1. If $1 \notin I$, then $M = S_1 \oplus (M_2 \oplus ... \oplus M_n)$. By modularity we get $S_1 \oplus S_2 = (S_1 \oplus S_2) \cap M = (S_1 \oplus S_2) \cap (S_2 \oplus M_1) = S_2 \oplus ((S_1 \oplus S_2) \cap M_1) = S_2 \oplus U$, where $U = (S_1 \oplus S_2) \cap M_1$. Therefore, $U \subseteq M_1, U \cong S_1 \cong M_1$. By our assumption, we must have $U = M_1$, and hence $S_1 \oplus S_2 = S_2 \oplus M_1 = M$.

Case 2. If $1 \in I$, then there is $k \neq 1$ such that $k = \{1, ..., n\} \backslash I$. By modularity we get $S_1 \oplus S_2 = S_2 \oplus V$, where $V = (S_1 \oplus S_2) \cap M_1$. Therefore, $V \subseteq M_1, V \cong S_1 \cong M_k$. By our assumption, we must have $V = M_1$, and hence $S_1 \oplus S_2 = S_2 \oplus M_1 = M$, as desired.

Proof of Theorem 1. (a), We show that M satisfies (C_3), i.e., for two direct summands S_1, S_2 of M with $S_1 \cap S_2 = 0$, $S_1 \oplus S_2$ is also a direct summand of M. By Lemma 1 we have $End(M_i)$, $i = 1, ..., n$ is a local ring. By Azumaya's Lemma (cf. [4, 12.6, 12.7]), we have $M = S_1 \oplus H = S_1 \oplus (\oplus_{i \in I} M_i) = (\oplus_{i \in J} M_i) \oplus (\oplus_{i \in I} M_i)$ (where $J = \{1, ..., n\} \backslash I$) and $M = S_2 \oplus K = S_2 \oplus (\oplus_{j \in E} M_j) = (\oplus_{j \in F} M_j) \oplus (\oplus_{j \in E} M_j)$ (where $F = \{1, ..., n\} \backslash E$). We imply $S_1 \cong \oplus_{i \in J} M_i$ and $S_2 \cong \oplus_{j \in F} M_j$. Suppose that $F = \{1, ..., k\}$. Let φ be isomorphism $\oplus_{i=1}^{k} M_j \longrightarrow S_2$. Set $X_j = \varphi(M_j)$, we have $X_j \cong M_j, S_2 = \oplus_{i=1}^{k} X_j$. By hypothesis $S_2 \subseteq^{\oplus} M$, we must have $X_j \subseteq^{\oplus} M$, $j = 1, ..., k$. We show that $S_1 \oplus S_2 = S_1 \oplus (X_1 \oplus ... \oplus X_k)$ is a direct summand of M.

We first prove a claim that $S_1 \oplus X_1$ is a direct summand of M. By Azumaya's Lemma (cf. [4, 12.6, 12.7]), we have $M = X_1 \oplus L = X_1 \oplus (\oplus_{s \in S} M_s) = M_\alpha \oplus (\oplus_{s \in S} M_s)$, with $S \subseteq \{1, ..., n\}$ such that $card(S) = n - 1$ and $\alpha = \{1, ..., n\} \backslash S$. Note that $card(S \cap I) \geq card(I) - 1 = m$. Suppose that $\{1, ..., m\} \subseteq (S \cap I)$, i.e., $M = (S_1 \oplus (M_1 \oplus ... \oplus M_m)) \oplus M_\beta = Z \oplus M_\beta$ with $\beta = I \backslash \{1, ..., m\}$ and $Z = S_1 \oplus (M_1 \oplus ... \oplus M_m)$. By M is a $(IEZ)-$module and $X_1 \cap S_1 = X_1 \cap (M_1 \oplus ... \oplus M_m) = S_1 \cap (M_1 \oplus ... \oplus M_m) = 0$, we have $Z \cap X_1 = 0$. By $Z, X_1 \subseteq^{\oplus} M$, $u - dim(Z) = n - 1$, $u - dim(X_1) = 1$, i.e., $u - dim(Z) + u - dim(X_1) = n$ and by Lemma 2 we have $M = Z \oplus X_1 = S_1 \oplus (M_1 \oplus ... \oplus M_m) \oplus X_1 = (S_1 \oplus X_1) \oplus (M_1 \oplus ... \oplus M_m)$. Therefore, $S_1 \oplus X_1 \subseteq^{\oplus} M$.

By induction we have $S_1 \oplus S_2 = S_1 \oplus (X_1 \oplus ... \oplus X_k) = (S_1 \oplus X_1 \oplus ... \oplus X_{k-1}) \oplus X_k$ is a direct summand of M, as desired.

(b), The implication $(i) \Longrightarrow (ii)$ is clear .

$(ii) \Longrightarrow (i)$. We show that M satisfies (C_2), i.e., for two submodules X, Y of M, with $X \cong Y$ and $Y \subseteq^{\oplus} M$, X is also a direct summand of M.

Note that, since $u - dim(M) = n$, we have $u - dim(Y) = 0, 1, ..., n$, the following case is trival: $u - dim(Y) = 0$.

If $u - dim(Y) = 1, ..., n$. By Azumaya's Lemma (cf. [4, 12.6, 12.7]) $X \cong Y \cong \oplus_{i \in I} M_i, I \subseteq \{1, ..., n\}$. Let φ be isomorphism $\oplus_{i \in I} M_i \longrightarrow X$. Set $X_i = \varphi(M_i)$, thus $X_i \cong M_i$ for any $i \in I$. By hypothesis (ii), we have $X_i \subseteq^{\oplus} M, i \in I$. Since $X = \oplus_{i \in I} X_i$ and X satisfies (C_3), thus $X \subseteq^{\oplus} M$, proving (i).

Lemma 3. *Let $M = M_1 \oplus ... \oplus M_n$, with all M_i uniform. Suppose that M is a nonsingular $(IEZ)-$module. Then M is a $CS-$module.*

Proof. We prove that each uniform closed submodule of M is a direct summand of M. Let A be a uniform closed submodule of M. Set $X_i = A \cap M_i$, $i = 1, ..., n$. Suppose that $X_i = 0$ for any $i = 1, ..., n$. By hypothesis, M is $(IEZ)-$module, we have $A = A \cap M = A \cap (M_1 \oplus ... \oplus M_n) = 0$, a contradiction. Therefore, there is a $X_j \neq 0$, i.e., $A \cap M_j \neq 0$. By property A and M_j are uniform submodules we have $A \cap M_j \subseteq^e A$ and $A \cap M_j \subseteq^e M_j$. By A and M_j are closure of $A \cap M_j$, M is a nonsingular module, we have $A = M_j \subseteq^{\oplus} M$. This implies that M is uniform - extending.

Since M has finite uniform dimension and by [1, Corollary 7.8], M is extending module, as desired.

Proof of Theorem 2. By Lemma 3, M is a $CS-$module. We show that M satisfies (C_2). By Theorem 1, we prove that if $X \subseteq M, X \cong M_i$ (with $i \in \{1, ..., n\}$), then $X \subseteq^{\oplus} M$.

Set X^* is a closure of X in M. Since M_i is a uniform module, thus X is also uniform. Therefore X^* is a uniform closed module. We imply X^* is a direct summand of M. We have $X^* = M_j$, thus $X \subseteq M_j$.

If $X \subseteq M_j, X \neq M_j$ then $X \subseteq J(M_j)$. Hence $M_i \cong X \subseteq J(M_j)$, a contradiction. We have $X = M_j \subseteq^{\oplus} M$, as desired.

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