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Some results on countably Σ–uniform - extending modules

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Abstract: A module M is called a uniform extending if every uniform submodule of M is essential in a direct summand of M. A module M is called a countably $\Sigma-$ uniform extending if $M^{(\mathbf{N})}$ is uniform extending. In this paper, we discuss the question of when a countably $\Sigma-$ uniform extending module is $\Sigma-$ quasi - injective? We also characterize QF rings by the class of countably $\Sigma-$ uniform extending modules.

1. Introduction

Throughout this note, all rings are associative with identity and all modules are unital right modules. The Jacobson radical and the injective hull of M are denoted by $J(M)$ and $E(M)$. If the composition length of a module M is finite, then we denote its length by $l(M)$.

For a module M consider the following conditions:

(C_1) Every submodule of M is essential in a direct summand of M.

(C_2) Every submodule isomorphic to a direct summand of M is itself a direct summand.

(C_3) If A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M.

Call a module M a CS - module or an extending module if it satisfies the condition (C_1); a continuous module if it satisfies (C_1) and (C_2), and a quasi-continuous if it satisfies (C_1) and (C_3). We now consider a weaker form of CS - modules. A module M is called a uniform extending if every uniform submodule of M is essential in a direct summand of M. We have the following implications:

Injective $\Rightarrow$ quasi - injective $\Rightarrow$ continuous $\Rightarrow$ quasi - continuous $\Rightarrow$ CS $\Rightarrow$ uniform extending.

$(C_2) \Rightarrow (C_3)$

We refer to [1] and [2] for background on CS and (quasi-)continuous modules.

A module M is called a (countably) $\Sigma-$uniform extending (CS, quasi - injective, injective) module if $M^{(A)}$ (respectively, $M^{(\mathbf{N})}$) is uniform extending (CS, quasi - injective, injective) for any set A. Note that $\mathbf{N}$ denotes the set of all natural numbers.

In this paper, we discuss the question of when a countably $\Sigma-$ uniform extending module is $\Sigma-$

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quasi - injective? We also characterize QF rings by the class of countably $\Sigma-$ uniform extending modules.

2. Introduction

Lemma 2.1. *Let $M = \oplus_{i\in I} M_i$ be a continuous module where each M_i is uniform. Then the following conditions are equivalent:*

(i) M is countably $\Sigma-$uniform extending,

(ii) M is $\Sigma-$ quasi - injective.

By Lemma 2.1, if M is a module with finite right uniform dimension such that $M \oplus M$ satisfies (C_3), then we have:

Proposition 2.2 *Let M be a module with finite right uniform dimension such that $M \oplus M$ satisfies (C_3). Then M is countably $\Sigma-$uniform extending if and only if M is $\Sigma-$quasi - injective.*

Proof. If M is countably $\Sigma-$uniform extending, then $M \oplus M$ is uniform extending. Since $M \oplus M$ has finite uniform dimension, $M \oplus M$ is CS. By $M \oplus M$ has (C_3), hence $M \oplus M$ is quasi - continuous. This implies that M is quasi - injective. Thus M is continuous module. Since M has finite uniform dimension, thus $M = U_1 \oplus ... \oplus U_n$ with U_i is uniform. By M is countably $\Sigma-$ uniform extending and by Lemma 2.1, M is $\Sigma-$ quasi - injective.

If M is $\Sigma-$ quasi - injective then M is countably $\Sigma-$uniform extending, is clear.

Corollary 2.3. *For $M = M_1 \oplus ... \oplus M_n$ is a direct sum of uniform local modules M_i such that M_i does not embed in $J(M_j)$ for any $i, j = 1, ..., n$ the following conditions are equivalent :*

(a) M is $\Sigma-$quasi - injective;

(b) M is countably $\Sigma-$uniform - extending.

Proof. The implications $(a) \Longrightarrow (b)$ is clear.

$(b) \Longrightarrow (a)$. By (b), $M \oplus M$ is extending module. By [4, Lemma 1.1], $M_i \oplus M_j$ has (C_3), hence $M_i \oplus M_j$ is quasi - continuous. By [5, Corollary 11], $M \oplus M$ is quasi - continuous. By Proposition 2.2, we have (a).

By Lemma 2.1 and Corollary 2.3, we characterized properties QF of a semiperfect ring by class countably $\Sigma-$uniform extending modules.

Corollary 2.4. *Let R be a semiperfect ring with $R = e_1R \oplus ... \oplus e_nR$ where each e_iR is a local right and $\{e_i\}_{i=1}^n$ is an orthogonal system of idempotents. Moreover assume that each e_jR is not embedable in any e_jJ $(i, j = 1, 2, ..., n)$. the following conditions are equivalent:*

(a) R is QF - ring;

(b) R_R is $\Sigma-$injective;

(c) R_R is countably $\Sigma-$uniform - extending.

Proof. $(a) \Longleftrightarrow (b)$, is clear.

$(b) \Longleftrightarrow (c)$, by Corollary 2.3.

Proposition 2.5. *Let R be a right continuous semiperfect ring, the following conditions are equivalent:*

(a) R is QF - ring;

(b) R_R is $\Sigma-$injective;

(c) R_R is countably $\Sigma-$uniform - extending.

Proof. $(a) \Longleftrightarrow (b)$, $(b) \Rightarrow (c)$ are clear.

$(c) \Rightarrow (b)$. Write $R_R = R_1 \oplus ... \oplus R_n$ where each R_i is unifom. Since R_R is right continuous, countably $\Sigma-$uniform extending, thus R_R is $\Sigma-$quasi - injective (by Lemma 2.1). Hence R_R is $\Sigma-$injective, proving (b).

Let $M = \bigoplus_{i\in I} U_i$, with all U_i uniform. We give properties of a closed submodule of M.

Lemma 2.6. ([6, Lemma 1]) *Let $\{U_i, \forall i \in I\}$ be a family of uniform modules. Set $M = \bigoplus_{i\in I} U_i$. If A is a closed submodule of M, then there is a subset F of I, such that $A \bigoplus(\bigoplus_{i\in F} U_i) \subseteq^e M$.*

By Lemma 2.1 and Lemma 2.6, we have:

Theorem 2.7. *Let $M = \bigoplus_{i\in I} U_i$ where each U_i is uniform. Assume that M is countably $\Sigma-$uniform - extending. Then the following conditions are equivalent:*

(i) M is $\Sigma-$ quasi - injective;

(ii) M satifies (C_2);

(iii) M satifies (C_3) and if $X \subseteq M, X \cong \bigoplus_{i\in J} U_i$ (with $J \subset I$) then $X \subseteq^{\oplus} M$.

Proof. The implications $(i) \Longrightarrow (ii)$ and $(ii) \Longrightarrow (iii)$ are clear.

$(iii) \Rightarrow (i)$. We show that M satisfies (C_2), i.e., for two submodules X, Y of M, with $X \cong Y$ and $Y \subseteq^{\oplus} M$, X is also a direct summand of M. Note that Y is a closed submodule of M. By Lemma 2.6, there is a subset F of I such that: $Y \bigoplus(\bigoplus_{i\in F} U_i) \subset^e M$. By hypothesis, $Y, \bigoplus_{i\in F} U_i \subseteq^{\oplus} M$ and M satifies (C_3), we have $M = Y \bigoplus(\bigoplus_{i\in F} U_i)$. If $F = I$ then $X = Y = 0$. Thus $X \subseteq^{\oplus} M$. If $F \neq I$, set $J = I\backslash F$, and we have $M = (\bigoplus_{i\in J} U_i)\bigoplus(\bigoplus_{i\in F} U_i)$. Thus $X \cong Y \cong M/\bigoplus_{i\in F} U_i \cong \bigoplus_{i\in J} U_i$. By hypothesis (iii), $X \subseteq^{\oplus} M$, as required.

Finally, we show that M is an extending module. Let us consider A is a closed submodule of M. By hypothesis A is a closed submodule of M and by Lemma 2.6, there is a submodule V_1 of M such that $V_1 = \bigoplus_{i\in F} U_i$, where $F \subset I$ satisfying: $A \bigoplus(\bigoplus_{i\in F} U_i) \subset^e M$. Set $V_2 = \bigoplus_{i\in K} U_i$ with $K = I\backslash F$. Let p_1, p_2 be the projection of M onto V_1 and V_2, then $p_2 \mid_A$ is a monomorphism (because $A \cap V_1 = 0$). Let $h = p_1(p_2 \mid_A)^{-1}$ be the homomorphism $p_2(A) \longrightarrow V_1$. We then have $A = \{x + h(x) \mid x \in p_2(A)\}$. Next, we aim to show next that h cannot be extended in V_2.

Suppose that h: $B \longrightarrow V_1$, where $p_2(A) \subseteq B \subseteq V_2$, is an extending of h in V_2. Set $C = \{x+h(x) \mid x \in B\}$, we have $A \oplus V_1 \subseteq^e M$, $p_2(A) = p_2(A \bigoplus V_1) \subseteq^e p_2(M) = V_2$. Hence $p_2(A) \subseteq^e B \subseteq V_2$, and thus $A \subseteq^e C$. Since A is a closed submodule, we have $A = C$, $p_2(A) = B$. Thus h= h.

Let us consider $k \in K$, set $X_k = U_k \cap p_2(A)$. We can see that $X_k \neq 0, \forall k \in K$. Therefore X_k is uniform module. Set $A_k = \{x + h(x) \mid x \in X_k\}$, we have $X_k \cong A_k$ and A_k is a uniform submodule of A. Suppose that $A_k \subseteq^e P \subseteq U_k \oplus V_1$. Since $A_k \cap V_1 = 0$, we have $P \cap V_1 = 0$, and thus $p_2 \mid_P$ is a monomorphism. Set $h_k = h \mid_{p_2(A_k)}$. Because h cannot be extended, we see that h_k cannot too. Set $\lambda_k = p_1(p_2 \mid_P)^{-1} : p_2(M) \longrightarrow V_1$. Thus λ_k is an extending of h_k and hence $p_2(P) = p_2(A_k)$. Since $p_2 \mid_P$ is a monomorphism and $A_k \subseteq^e P$. It follow that $A_k = P$.

Hence A_k is a uniform closed submodule and M is a uniform extending (because M is countably $\Sigma-$uniform - extending). Thus $A_k \subseteq^{\oplus} M$. Since A_k is a closed submodule of M and by Lemma 2.6, there is a submodule V_3 of M such that $V_3 = \bigoplus_{i\in L} U_i$, where $L \subset I$ satisfying $A_k \bigoplus V_3 \subseteq^e M$. Since $A_k \subseteq^{\oplus} M$, $V_3 \subseteq^{\oplus} M$ and M satifies (C_3), we have $A_k \oplus V_3 \subseteq^{\oplus} M$, $A_k \oplus V_3 = M$. Suppose that $V_4 = \oplus_{i\in J} U_i$ where $J = I\backslash L$. Then $M = A_k \oplus V_3 = V_4 \oplus V_3$, and we have $A_k \cong M/V_3 = V_4 \oplus V_3/V_3 \cong V_4$. Because A_k is a uniform module, $\mid J \mid= 1$, i.e., $A_k \cong U_j (j \in I)$ we infer that $X_k \cong A_k \cong U_j$. Therefore $X_k \subseteq^{\oplus} M$. But $X_k \subseteq U_k \subseteq^{\oplus} M$ and hence $X_k = U_k$, for all $k \in F$. Therefore $p_2(A) = V_2$, and we have $A \cong V_2$. Note that $A \cong V_2 = \bigoplus_{i\in K} U_i$, we must have $A \subseteq^{\oplus} M$. Therefore M is an extending module. But M satisfies (C_2), and thus M is a continuous module. Therefore by Lemma 2.1, proving (i).

By Lemma 2.1 and Theorem 2.7, we characterized QF property of a ring with finite right uniform dimension by the class countably $\Sigma-$ uniform extending modules.

Theorem 2.8. *Let R be a ring with finite right uniform dimension such that $R_R^{(\mathbf{N})}$ is uniform extending, the following conditions are equivalent:*

(a) R_R is self - injective;

(b) $(R \oplus R)_R$ satisfies (C_3);

(c) R_R satisfies (C_2);

(d) R_R is $\Sigma-$injective;

(e) R is QF - ring.

*Proof.*The implications $(a) \Rightarrow (b)$, $(a) \Rightarrow (c)$, $(d) \Rightarrow (a)$ and $(d) \Longleftrightarrow (e)$ are clear.

$(b) \Rightarrow (d)$. Because R_R has finite uniform dimension, therefore $(R \oplus R)_R$ has finite uniform dimension. But $R_R^{(\mathbf{N})}$ is a uniform extending, thus$(R \oplus R)_R$ is a uniform extending, and hence $(R \oplus R)_R$ is extending. Because $(R \oplus R)_R$ has (C_3), thus $(R \oplus R)_R$ is a quasi - continuous modules. Therefore, R_R is quasi - injective, and thus $R_R = R_1 \oplus ... \oplus R_n$ where each R_i is uniform. By R_R is continuous and $R_R^{(\mathbf{N})}$ is uniform extending we have R_R is $\Sigma-$ quasi - injective (by Lemma 2.1). Hence R_R is $\Sigma-$injective, proving (d).

$(c) \Rightarrow (d)$. By R_R has finite uniform dimension, thus $R_R = R_1 \oplus ... \oplus R_n$ with R_i is uniform. By Theorem 2.7, R_R is $\Sigma-$injective, proving (d).

A ring R is called a right CS if R_R is CS module. By Theorem 2.8, we have.

Corollary 2.9. *Let R be a right CS ring with finite right uniform dimension such that every extending right $R-$module is countably $\Sigma-$uniform - extending. If $(R \oplus R)_R$ satisfies (C_3) then R is QF ring.*

Proof. Since R_R is CS, thus $R_R^{(\mathbf{N})}$ is uniform extending. By Theorem 2.8, R_R is $\Sigma-$injective. Therefore, R is QF ring.

Lemma 2.10. *Let U_1, U_2 be uniform modules such that $l(U_1) = l(U_2) < \infty$. Set $U = U_1 \oplus U_2$. Then U satisfies (C_3).*

Proof.(a) By [7], $End(U_1)$ and $End(U_2)$ are local rings. We show that U satisfies (C_3), i.e., for two direct summands S_1, S_2 of U with $S_1 \cap S_2 = 0$, $S_1 \oplus S_2$ is also a direct summand of U. Note that, since u - dim(U) = 2, the following case is trivial:

If one of the $S_i's$ has uniform dimension 2, the other is zero.

Hence we consider the case that both S_1, S_2 are uniform. Write $U = S_2 \oplus K$. By Azumaya's Lemma (cf. [8, 12.6, 12.7]), either $S_2 \oplus K = S_2 \oplus U_i$, or $S_2 \oplus K = S_2 \oplus U_j$. Since i and j can interchange with each other, we need only to consider one of the two possibilities. Let us consider the case $U = S_2 \oplus K = S_2 \oplus U_1 = U_1 \oplus U_2$. Then it follows $S_2 \cong U_2$. Write $U = S_1 \oplus H$. Then either $U = S_1 \oplus H = S_1 \oplus U_1$ or $S_1 \oplus H = S_1 \oplus U_2$.

If $U = S_1 \oplus H = S_1 \oplus U_1$, then by modularity we get $S_1 \oplus S_2 = S_1 \oplus X$ where $X = (S_1 \oplus S_2) \cap U_1$. From here we get $X \cong S_2 \cong U_2$. Since $l(U_1) = l(U_2) = l(X)$, we have $U_1 = X$, and hence $S_1 \oplus S_2 = S_1 \oplus U_1 = U$.

If $U = S_1 \oplus H = S_1 \oplus U_2$, then by modularity we get $S_1 \oplus S_2 = S_1 \oplus V$ where $V = (S_1 \oplus S_2) \cap U_2$. From here we get $V \cong S_2 \cong U_2$. Since $l(U_2) = l(V)$, we have $U_2 = V$, and hence $S_1 \oplus S_2 = S_1 \oplus U_2 = U$.

Thus U satisfies (C_3), as desired.

By Lemma 2.10 and Proposition 2.2, we have:

Proposition 2.11. *For $M = M_1 \oplus ... \oplus M_n$ is a direct sum of uniform modules M_i such that*

$l(M_1) = l(M_2) = ... = l(M_n) < \infty$, the following conditions are equivalent :

(a) M is $\Sigma-$quasi - injective;

(b) M is countably $\Sigma-$uniform - extending.

Proof. $(a) \Longrightarrow (b)$ is clear.

$(b) \Longrightarrow (a)$. By (b) and by Lemma 2.10, $M_i \oplus M_j$ is quasi - continuous. By [5, Corollary 11], $M \oplus M$ is quasi - continuous. By Proposition 2, we have (a).

Lemma 2.12. *Let R be a ring with $R = e_1R \oplus ... \oplus e_nR$ where each e_iR is a uniform right ideal and $\{e_i\}_1^n$ is a system of idempotents. Moreover, assume that $l(e_1R) = l(e_2R) = ... = l(e_nR) < \infty$. Then R is right self - injective if and only if $(R \oplus R)_R$ is CS.*

Proof. By Lemma 2.10 and by [2, 2.10].

By Lemma 2.1 and Lemma 2.12, we have:

Proposition 2.13. *Let R be a ring with $R = e_1R \oplus ... \oplus e_nR$ where each e_iR is a uniform right ideal and $\{e_i\}_1^n$ is a system of idempotents. Moreover, assume that $l(e_1R) = l(e_2R) = ... = l(e_nR) < \infty$, the following conditions are equivalent:*

(a) R is QF - ring;

(b) R_R is $\Sigma-$injective;

(c) R_R is countably $\Sigma-$uniform - extending.

Proof. $(a) \Longleftrightarrow (b)$, $(b) \Rightarrow (c)$ are clear.

$(c) \Rightarrow (b)$. By $(R \oplus R)_R$ has finite uniform dimension and by (c), $(R \oplus R)_R$ is CS. By Lemma 2.12, R_R is a continuous module. By Lemma 2.1, R_R is $\Sigma-$quasi - injective. Hence R_R is $\Sigma-$injective, proving (b).

Proposition 2.14. *Let R be a right Noetherian ring and M a right $R-$ module such that $M = \oplus_{i \in I} M_i$ is a direct sum of uniform submodules M_i. Suppose that $M \oplus M$ satisfies (C_3), the following conditions are equivalent:*

(a) M is $\Sigma-$quasi - injective;

(b) M is countably $\Sigma-$uniform - extending.

Proof.$(a) \Longrightarrow (b)$ is clear.

$(b) \Longrightarrow (a)$. By $M_i \oplus M_j$ is direct summand of $M \oplus M$ and by hypothesis (b), thus $M_i \oplus M_j$ is quasi - continuous. Hence M_i is M_j- injective for any $i, j \in I$. Note that R is a right Noetherian ring, thus M is quasi - injective (see [2, Proposition 1.18]), i.e., M satifies (C_2). By Theorem 2.7, we have (a).

Proposition 2.15. *Let R be a right Noetherian ring and M a right $R-$ module such that $M = \oplus_{i \in I} M_i$ is a direct sum of uniform local submodules M_i. Suppose that M_i does not embed in $J(M_j)$ for any $i, j \in I$, the following conditions are equivalent:*

(a) M is $\Sigma-$quasi - injective;

(b) M is countably $\Sigma-$uniform - extending.

Proof.$(a) \Longrightarrow (b)$ is clear.

$(b) \Longrightarrow (a)$. By (b), $M \oplus M$ is uniform - extending. Hence $M_i \oplus M_j$ is CS for any $i, j \in I$. By [4, Lemma 1.1], $M_i \oplus M_j$ is quasi - continuous, thus M_i is M_j- injective for any $i, j \in I$. Therefore M is quasi - injective (see [2, Proposition 1.18]), i.e., M satifies (C_2). By Theorem 2.7, we have (a).

Proposition 2.16. *Let R be a right Noetherian ring and M a right $R-$ module such that $M = \oplus_{i \in I} M_i$ is a direct sum of uniform submodules M_i. Suppose that $l(M_i) = n < \infty$ for any $i \in I$, the following conditions are equivalent:*

(a) M is $\Sigma-$quasi - injective;

(b) M is countably $\Sigma-$uniform - extending.

Proof. By Lemma 2.10, Theorem 2.7 and [2, Proposition 1.18].

Proposition 2.17. *There exists a right Noetherian ring R and a right $R-$ module countably $\Sigma-$uniform - extending M such that $M = \oplus_{i\in I} M_i$ is a direct sum of uniform submodules M_i, M satisfies (C_3) but is not $\Sigma-$quasi - injective.*

*Proof.*Let $R = \mathbf{Z}$ be the ring of integer numbers, then R is a right (and left) Noetherian ring, and let $M = R_1 \oplus R_2 \oplus ... \oplus R_n$ with $R_1 = R_2 = ... = R_n = R_R = \mathbf{Z_Z}$. We have $M^{(\mathbf{N})} = \oplus_{i=1}^{\infty} M_i$ with $M_i = M$, we imply $M = (R_1 \oplus ... \oplus R_n)^{(\mathbf{N})} = \mathbf{Z}^{(\mathbf{N})}$. By [1, page 56], M is countably $\Sigma-$uniform - extending. Since $R_i = \mathbf{Z_Z}$ is a uniform module for any $i = 1, 2, ..., n$ thus M is a finite direct sum of uniform submodules. But also by [1, page 56], M is not countably $\Sigma-$ CS module. Therefore, M is not countably $\Sigma-$quasi - injective, i.e., M is not $\Sigma-$quasi - injective. If $n = 1$, then M satisfies (C_3), as desired.

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