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The extreme value of local dimension of convolution of the cantor measure

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Abstract. Let μ be the $m-$fold convolution of the standard Cantor measure and $\underline{\alpha}_m$ be the lower extreme value of the local dimension of the measure μ. The values of $\underline{\alpha}_m$ for $m = 2, 3, 4$ were showed in [4] and [5]. In this paper, we show that

$$\underline{\alpha}_5 = \left|\frac{\log\left[\frac{2}{3.2^5}\left(\sqrt{145}\cos\left(\frac{\arccos\frac{427}{59\sqrt{145}}}{3}\right)+5\right)\right]}{\log 3}\right| \approx 0.972638.$$

This values was estimated by P. Shmerkin in [5], but it has not been proved.
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1. Introduction

Let $\{S_j\}_{j=1}^m$ be contractive similitudes on $\mathbb{R}^d$ and $\{p_j\}_j^m (0 \leqslant p_j \leqslant 1, \sum\limits_{j=1}^m p_j = 1)$ be a set of probability weights. Then, there exists a unique probability measure μ satisfying

$$\mu(A) = \sum_{j=1}^m p_j \mu(S_j^{-1}(A))$$

for all Borel measurable sets A (see [1]). We call μ a *self-similar measure* and $\{S_j\}_{j=1}^m$ a *system iterated functions*.

When $S_1, ..., S_m$ are similarities with equal contraction ratio $\rho \in (0, 1)$ on $\mathbb{R}$, i.e., $S_j(x) = \rho(x + b_j),\ b_j \in \mathbb{R}$ for $j = 1, ..., m$, the self-similar measure μ can be seen as follows: Let $X_0, X_1, ...$ be a sequence of independent identically distributed random variables each taking real values $b_1, ..., b_m$ with probability $p_1, ..., p_m$ respectively. We define a random variable $S = \sum\limits_{i=1}^{\infty} \rho^i X_i$, then the probability measure μ_ρ induced by S :

$$\mu_\rho(A) = P\{\omega : S(\omega) \in A\}$$

is called a *fractal measure* and $\mu_\rho \equiv \mu$ (see [2]).

Let ν be the standard Cantor measure, then ν can be considered to be generated by the two maps $S_i(x) = \frac{1}{3}x + \frac{2}{3}i,\ i = 0, 1$ with weight $\frac{1}{2}$ on each S_i. Then the attractor of this system

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iterated functions is the standard Cantor set C, i.e., $C = S_0(C) \cup S_1(C)$. Let $\mu = \nu * ... * \nu$ be the $m-$fold convolution of the standard Cantor measure. For $m \geq 3$, this measure does not satisfy the open set condition (see [2]), so the studying the local dimension of this measure in this case is very difficult. Another convenient way to look at μ is as the distribution of the random sum, i.e., μ can be obtained in the following way: Let X be a random variable taking values $\{0, 1, ..., m\}$ with probality $p_i = P(X = i) = \frac{C_m^i}{2^m}, i = 0, 1, ..., m$ and let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variable with the same distribution as X. Let $S = \sum\limits_{j=1}^{\infty} 3^{-j} X_j,\ S_n = \sum\limits_{j=1}^{n} 3^{-j} X_j$ and $\mu,\ \mu_n$ be the distribution measure of $S,\ S_n$ respectively. It is well known that μ is either singular or absolutely continuous (see [2]).

Recall that let μ be a probability measure on $\mathbb{R}$. For $s \in \text{supp } \mu$, the local dimension of μ at s is denoted by $\alpha(s)$ and defined by

$$\alpha(s) = \lim_{h \to 0^+} \frac{\log \mu(B_h(s))}{\log h}$$

if the limit exists. Otherwise, let $\overline{\alpha}(s)$ and $\underline{\alpha}(s)$ denote the upper and lower dimension by taking the upper and lower limits respectively. Let $E = \{\alpha(s) : s \in \text{supp } \mu\}$ be the set of the attainable local dimensions of the measure μ and for each $m = 2, 3, ...$, put

$$\underline{\alpha}_m = \inf\{\underline{\alpha}(s) : s \in \text{supp } \mu\};$$

$$\overline{\alpha}_m = \sup\{\overline{\alpha}(s) : s \in \text{supp } \mu\}.$$

It is showed in [4] that $\overline{\alpha}_m = \frac{m \log 2}{\log 3}$ is an isolated point of E for all $m = 2, 3, ...$ and

$$\underline{\alpha}_m = \frac{\log 2}{\log 3} \approx 0.63093 \text{ if } m = 2;$$

$$\underline{\alpha}_m = \frac{3 \log 2}{\log 3} - 1 \approx 0.89278 \text{ if } m = 3 \text{ or } 4.$$

This results were proved by using combinatoric, it depends on some careful counting of the multiple representations of $s = \sum\limits_{j=1}^{\infty} 3^{-j} x_j, x_j = 0, ..., m$, and the associated probability. After that, in [5], Pablo Shmerkin showed the $\underline{\alpha}_m$ for $m = 2, 3, 4$ by the other way. He used the spectral radius of matrixes to define his results. He said that the identifying formulae for $\underline{\alpha}_m$ for $m \geq 5$ was a difficult problem, and he only estimated the values of $\underline{\alpha}_m$ for $5 \leqslant m \leqslant 10$.

Now, in this paper, we are interested in the identifying $\underline{\alpha}_m$ for $m = 5$ and we show that our result coincides with Pablo Shmerkin's estimate. We have

2. Main result

Main Theorem. *Let μ be the $5-$fold convolution of the standard Cantor measure, then the lower extreme value of the local dimension of μ is*

$$\underline{\alpha}_5 = \left|\frac{\log \left[\frac{2}{3.2^5}\left(\sqrt{145}\cos\left(\frac{\arccos \frac{427}{59\sqrt{145}}}{3}\right) + 5\right)\right]}{\log 3}\right| \approx 0.972638.$$

The proof of our Maim Theorem is divided in to two steps. In Section 2.1 we will give some notations and primary results. The Main Theorem is proved in Section 2.2.

2.1 Notations and Primary Results

Let ν be the standard Cantor measure and $\mu = \nu * ... * \nu$ ($m-$fold). Then, by similar proof as the Lemma 4.4 in [5], we have

Proposition 1. *Let ν be the standard Cantor measure, i.e., ν is induced by the two maps $S_i(x) = \frac{1}{3}x + \frac{2}{3}i,\ i = 0, 1$ with weigh $\frac{1}{2}$ on each S_i. Then its $m-$fold convolution $\mu = \nu * ... * \nu$ is generated by $S_i(x) = \frac{1}{3}x + \frac{2}{3}i$ with weight $\frac{C_m^i}{2^m}$ on with S_i for $i = 0, 1, ..., m$.*

Proposition 2 ([4]). *Let $m \geq 2$, then $\alpha(s) = \lim\limits_{n\to\infty} |\frac{\log \mu_n(s_n)}{n \log 3}|$ provided that the limit exists. Otherwise, we can replace $\alpha(s)$ by $\overline{\alpha}(s)$ and $\underline{\alpha}(s)$ and consider the upper and the lower limits.*

Put $D = \{0, 1, ..., 5\}$ and for each $n \in \mathbb{N}$ we denote

$$D^n = \{(x_1, ..., x_n) : x_i \in D\} D^\infty = \{(x_1, x_2, ...) : x_i \in D\}.$$

For $(x_1, ..., x_n) \in D^n$, put

$$\langle(x_1, ..., x_n)\rangle = \{(y_1, ..., y_n) \in D^n : \sum_{i=1}^{n} 3^{-i} y_i = \sum_{i=1}^{n} 3^{-i} x_i\}.$$

If $(z_1, ..., z_n) \in \langle(x_1, ..., x_n)\rangle$, then we denote $(z_1, ..., z_n) \sim (x_1, ..., x_n)$. Clearly that if $(z_1, ..., z_n) \sim (x_1, ..., x_n)$ and $(z_{n+1}, ..., z_m) \sim (x_{n+1}, ..., x_m)$ then

$$(z_1, ..., z_m) \sim (x_1, ..., x_m). \tag{1}$$

We denote

$$\langle(x_1, ..., x_n, x)\rangle = \{(y_1, ..., y_n, x) : (y_1, ..., y_n) \in \langle(x_1, ..., x_n)\rangle\}.$$

The following lemma will be used frequently in this paper.

Lemma 1. *Let $s_n = \sum\limits_{j=1}^{n} 3^{-j} x_j,\ s'_n = \sum\limits_{j=1}^{n} 3^{-j} x'_j$ be two points in supp μ_n. If $s_n = s'_n$ then $x_n \equiv x'_n$ (mod 3).*

Proposition 3. *Let $x = (x_1, x_2, ...) = (2, 3, 2, 3, ...) \in D^\infty$, we have*

i) If n is even then $(y_1, ..., y_n) \in \langle(x_1, ..., x_n)\rangle = \langle(2, 3, ..., 2, 3)\rangle$ iff

$$(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-1}, 3)\rangle \text{ or } (y_1, ..., y_n) \in \langle(x_1, ..., x_{n-2}, x_{n-2}, 0)\rangle.$$

ii) If n is odd then $(y_1, ..., y_n) \in \langle(x_1, ..., x_n)\rangle = \langle(2, 3, ..., 2, 3, 2)\rangle$ iff

$$(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-1}, 2)\rangle \text{ or } (y_1, ..., y_n) \in \langle(x_1, ..., x_{n-2}, x_{n-2}, 5)\rangle.$$

Proof.

i) The case n is even.

If $(y_1, ..., y_n) \in \langle(x_1, ..., x_n)\rangle = \langle(2, 3, ..., 2, 3)\rangle$ then we have

$$(y_1 - 2)3^{n-1} + (y_2 - 3)3^{n-2} + ... + (y_{n-1} - 2)3 + (y_n - 3) = 0. \tag{2}$$

Therefore, $y_n - 3 \equiv 0$ (mod 3). Since $y_n \in D$, we have $y_n = 3$ or $y_n = 0$.

a) If $y_n = 3$ then $y_n - 3 = 0$. By (2) we have $\sum\limits_{j=1}^{n-1} 3^{-j} y_j = \sum\limits_{i=1}^{n-1} 3^{-i} x_i$. Hence, $(y_1, ..., y_{n-1}) \in \langle(x_1, ..., x_{n-1})\rangle$. By (1) we have $(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-1}, 3)\rangle$.

b) If $y_n = 0$ then $y_n - 3 = -3$. By (2) we have

$$(y_1 - 2)3^{n-2} + (y_2 - 3)3^{n-3} + ... + (y_{n-2} - 3)3 + (y_{n-1} - 3) = 0.$$

Hence, $(y_1, ..., y_{n-2}, y_{n-1}) \in \langle(2, 3, ..., 2, 3, 3)\rangle = \langle(x_1, ..., x_{n-2}, x_{n-2})\rangle$. By (1) we have $(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-2}, x_{n-2}, 0)\rangle$.

Conveserly, if $(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-1}, 3)\rangle$, then we have

$$(y_1, ..., y_n) \in \langle(x_1, ..., x_n)\rangle.$$

So we consider the case $(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-2}, x_{n-2}, 0)\rangle$. Then we have $y_n = 0$ and $(y_1, ..., y_{n-1}) \in \langle(2, 3, ..., 2, 3, 3)\rangle$. We will show that $(y_1, ..., y_n) \in \langle(x_1, ..., x_n)\rangle$. In fact, since $(y_1, ..., y_{n-1}) \in \langle(2, 3, ..., 2, 3, 3)\rangle$, by Lemma 1 we have $y_{n-1} - 3 \equiv 0 \pmod 3$. This implies that $y_{n-1} = 3$ or $y_{n-1} = 0$.

a) If $y_{n-1} = 3$ then $y_{n-1} - 3 = 0$ and

$$(y_1 - 2)3^{n-2} + (y_2 - 3)3^{n-3} + ... + (y_{n-3} - 3)3 + (y_{n-2} - 3) = 0.$$

Therefore, $(y_1, ..., y_{n-2}) \sim (2, 3, ..., 2, 3) = (x_1, ..., x_{n-2})$ and $(y_{n-1}, y_n) = (3, 0)$. Since $(3, 0) \sim (2, 3)$, by (1) we have $(y_1, ..., y_n) \in \langle(x_1, ..., x_n)\rangle$.

b) If $y_{n-1} = 0$ then from $(y_1, ..., y_{n-1}) \sim (2, 3, ..., 2, 3, 3)$ we get

$$(y_1 - 2)3^{n-2} + (y_2 - 3)3^{n-3} + ... + (y_{n-2} - 3)3 - 3 = 0.$$

Hence,

$$(y_1 - 2)3^{n-2} + (y_2 - 3)3^{n-3} + ... + (y_{n-3} - 2)3 + y_{n-2} - 4 = 0. \quad (3)$$

Therefore, $y_{n-2} - 4 \equiv 0 \pmod 3$. Since $y_{n-2} \in D$, we have $y_{n-2} = 4$ or $y_{n-2} = 1$. We consider the two following cases.

Case 1. $y_{n-2} = 4$, then $(y_{n-2}, y_{n-1}, y_n) = (4, 0, 0)$ and $y_{n-2} - 4 = 0$. By (3) we have $(y_1, ..., y_{n-3}) \in \langle(2, 3, ..., 2, 3, 2)\rangle$. Since $(4, 0, 0) \sim (3, 2, 3)$, by (1) we have $(y_1, ..., y_n) = (y_1, ..., y_{n-3}, 4, 0, 0) \in \langle(2, 3, ..., 2, 3)\rangle = \langle(x_1, ..., x_n)\rangle$.

Case 2. $y_{n-2} = 1$, then $y_{n-2} - 4 = -3$ and $(y_{n-2}, y_{n-1}, y_n) = (4, 0, 0)$. From (3), we get

$$(y_1 - 2)3^{n-4} + (y_2 - 3)3^{n-3} + ... + (y_{n-4} - 3)3 + y_{n-3} - 3 = 0. \quad (4)$$

Therefore, $(y_1, ..., y_{n-3}) \in \langle(2, 3, ..., 2, 3, 3)\rangle$. By similar argument, we get $y_{n-3} = 0$ or $y_{n-3} = 3$.

+) If $y_{n-3} = 3$ then $(y_{n-3}, y_{n-2}, y_{n-1}, y_n) = (3, 1, 0, 0)$ and from (4) we get $(y_1, ..., y_{n-4}) \in \langle(2, 3, ..., 2, 3)\rangle$. Since $(3, 1, 0, 0) \sim (2, 3, 2, 3)$, we get $(y_1, ..., y_n) \in \langle(2, 3, ..., 2, 3)\rangle = \langle(x_1, ..., x_n)\rangle$.

+) If $y_{n-3} = 0$ then the form (4) is similar to the form (3). Thus, by repeating about argument we get the proof of the proposition in this case of n.

ii) The case n is odd.

Assume that $(y_1, ..., y_n) \in \langle(x_1, ..., x_n)\rangle = \langle(2, 3, ..., 2, 3, 2)\rangle$ then

$$(y_1 - 2)3^{n-1} + (y_2 - 3)3^{n-2} + ... + (y_{n-1} - 3)3 + y_n - 2 = 0. \quad (5)$$

This implies $y_n = 2$ or $y_n = 5$.

a) If $y_n = 2$ then from (5), we have

$$(y_1, ..., y_{n-1}) \in \langle(2, 3, ..., 2, 3)\rangle = \langle(x_1, ..., x_{n-1})\rangle.$$

This means

$$(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-1}, 2)\rangle.$$

b) If $y_n = 5$ then from (5), we have

$$(y_1 - 2)3^{n-2} + (y_2 - 3)3^{n-3} + ... + (y_{n-2} - 2)3 + y_{n-1} - 2 = 0.$$

Therefore, $(y_1, ..., y_{n-1}) \sim (2, 3, ..., 2, 3, 2, 2) = (x_1, ..., x_{n-2}, x_{n-2})$. This implies

$$(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-2}, x_{n-2}, 5)\rangle.$$

Conversely, if $(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-1}, 2)\rangle$ then we have immediately that $(y_1, ..., y_n) \in \langle(x_1, ..., x_n)\rangle$. So we consider the following case

$$(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-2}, x_{n-2}, 5)\rangle.$$

then we have $y_n = 5$ and

$$(y_1, ..., y_{n-1}) \in \langle(x_1, ..., x_{n-2}, x_{n-2})\rangle = \langle(2, 3, ..., 2, 3, 2, 2)\rangle.$$

We will prove that $(y_1, ..., y_n) \in \langle(x_1, ..., x_n)\rangle$.

In fact, since $(y_1, ..., y_{n-1}) \in \langle(2, 3, ..., 2, 3, 2, 2)\rangle$, we have

$$(y_1 - 2)3^{n-2} + (y_2 - 3)3^{n-3} + ... + (y_{n-2} - 2)3 + y_{n-1} - 2 = 0. \tag{6}$$

Therefore, $y_{n-1} = 2$ or $y_{n-1} = 5$.

a) If $y_{n-1} = 2$ then from (6), we have $(y_1, ..., y_{n-2}) \in \langle(2, 3, ..., 2, 3, 2)\rangle$ and $(y_{n-1}, y_n) = (2, 5)$. Since $(2, 5) \sim (3, 2)$, by (1) we have

$$(y_1, ..., y_n) \in \langle(2, 3, ..., 2, 3, 2)\rangle = \langle(x_1, ..., x_n)\rangle.$$

b) If $y_{n-1} = 5$ then from (6), we have

$$(y_1 - 2)3^{n-3} + (y_2 - 3)3^{n-4} + ... + (y_{n-3} - 3)3 + y_{n-2} - 1 = 0. \tag{7}$$

Therefore, $y_{n-2} = 1$ or $y_{n-2} = 4$.

b1) If $y_{n-2} = 1$ then from (7), we have $(y_1, ..., y_{n-3}) \in \langle(2, 3, ..., 2, 3)\rangle$ and $(y_{n-2}, y_{n-1}, y_n) = (1, 5, 5)$. Since $(1, 5, 5) \sim (2, 3, 2)$, by (1) we have

$$(y_1, ..., y_n) \in \langle(2, 3, ..., 2, 3, 2)\rangle = \langle(x_1, ..., x_n)\rangle.$$

b2) If $y_{n-2} = 4$ then from (7), we have $(y_1, ..., y_{n-3}) \in \langle(2, 3, ..., 2, 3, 2, 2)\rangle$ and $(y_{n-2}, y_{n-1}, y_n) = (4, 5, 5)$. Since $(4, 5, 5) \sim (5, 3, 2)$, by (1) we have $(y_1, ..., y_n) \in \langle(2, 3, ...2, 3, 5, 3, 2)\rangle$. Therefore, by repeating above argument for the case $y_{n-2} = 5$ and

$$(y_1, ..., y_{n-3}) \in \langle(x_1, ..., x_{n-2}, x_{n-2})\rangle = \langle(2, 3, ..., 2, 3, 2, 2)\rangle.$$

We have the assertion of the proposition.

From Proposition 3 we have the following corollary.

Corrolary 1. *Let* $x = (x_1, x_2, ...) = (2, 3, 2, 3, ...) \in D^\infty$. *For each* $n \in \mathbb{N}$, *put* $s_n = \sum_{i=1}^{n} 3^{-i} x_i$ *and* $s'_n = \sum_{i=1}^{n} 3^{-i} x'_i$, *where* $(x'_1, ..., x'_{n-1}, x'_n) = (x_1, ..., x_{n-1}, x_{n-1})$. *Then we have*

i) $\mu_1(s_1) = \mu_1(s'_1) = \frac{10}{2^5}, \quad \mu_2(s_2) = \frac{110}{2^{10}}, \quad \mu_2(s'_2) = \frac{105}{2^{10}}$.

ii) $\mu_n(s_n) = \frac{10}{2^5}\mu_{n-1}(s_{n-1}) + \frac{1}{2^5}\mu_{n-1}(s'_{n-1})$.

Proof. i) For $n = 1$ we have $\langle(x_1)\rangle = \langle(x'_1)\rangle = \{(2)\}$. Therefore,

$$\mu_1(s_1) = \mu_1(s'_1) = P(X_1 = 2) = \frac{10}{2^5}.$$

For $n = 2$ we have $\langle(x_1, x_2)\rangle = \{(2,3),\ (3,0)\}$ and $\langle(x'_1, x'_2)\rangle = \{(2,2),\ (1,5)\}$. Therefore,

$$\mu_2(s_2) = \frac{10}{2^5}.\frac{10}{2^5} + \frac{10}{2^5}.\frac{1}{2^5} = \frac{110}{2^{10}};$$

$$\mu_2(s'_2) = \frac{10}{2^5}.\frac{10}{2^5} + \frac{5}{2^5}.\frac{1}{2^5} = \frac{105}{2^{10}}.$$

ii) By Proposition 3, we have

a) If n is even then

$$\langle(x_1, ..., x_n)\rangle = \langle(x_1, ..., x_{n-1}, 3)\rangle \cup \langle(x'_1, ..., x'_{n-1}, 0)\rangle.$$

b) If n is odd then

$$\langle(x_1, ..., x_n)\rangle = \langle(x_1, ..., x_{n-1}, 2)\rangle \cup \langle(x'_1, ..., x'_{n-1}, 5)\rangle.$$

Therefore, for all $n \in \mathbb{N}$ we have

$$\mu_n(s_n) = P(X_n = 2)\mu_{n-1}(s_{n-1}) + P(X_n = 5)\mu_{n-1}(s'_{n-1})$$
$$= \frac{10}{2^5}\mu_{n-1}(s_{n-1}) + \frac{1}{2^5}\mu_{n-1}(s'_{n-1}).$$

The corollary is proved. □

To have the recurrence formula of $\mu_n(s_n)$, we need the following proposition.

Proposition 4. *Let $x = (x_1, x_2, ...) = (2,3,2,3,...) \in D^\infty$. For each $n \in \mathbb{N}$, put $(x'_1, ..., x'_n) = (x_1, ..., x_{n-1}, x_{n-1})$. Then we have*

i) If n is even then $(y_1, ..., y_n) \in \langle(x'_1, ..., x'_n)\rangle = \langle(2,3,...,2,3,2,2)\rangle$ iff

$$(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-1}, 2)\rangle \cup \langle(x_1, ..., x_{n-2}, 1, 5)\rangle \cup \langle(x'_1, ..., x'_{n-2}, 4, 5)\rangle.$$

ii) If n is odd then $(y_1, ..., y_n) \in \langle(x'_1, ..., x'_n)\rangle = \langle(2,3,...,2,3,3)\rangle$ iff

$$(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-1}, 3)\rangle \cup \langle(x_1, ..., x_{n-2}, 4, 0)\rangle \cup \langle(x'_1, ..., x'_{n-2}, 1, 0)\rangle.$$

Proof. i) The case n is even.

a) If $(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-1}, 2)\rangle$ then $y_n = 2$ and $(y_1, ..., y_{n-1}) \in \langle(x_1, ..., x_{n-1})\rangle$. Therefore, by (1) we have

$$(y_1, ..., y_n) \in \langle(2,3,...,2,3,2,2)\rangle = \langle(x'_1, ..., x'_n)\rangle.$$

b) If $(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-2}, 1, 5)\rangle$ then $y_n = 5,\ y_{n-1} = 1$ and

$$(y_1, ..., y_{n-2}) \in \langle(x_1, ..., x_{n-2})\rangle = \langle(2,3,...,2,3)\rangle.$$

Since $(1,5) \sim (2,2)$, by (1) we have

$$(y_1, ..., y_n) \in \langle(2,3,...,2,3,2,2)\rangle = \langle(x'_1, ..., x'_n)\rangle.$$

c) If $(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-2}, 4, 5)\rangle = \langle(2,3,...,2,3,2,2,4,5)\rangle$ then by (1) we have

$$(y_1, ..., y_n) \in \langle(2,3,...,2,3,2,2)\rangle = \langle(x'_1, ..., x'_n)\rangle$$

since $(2,2,4,5) \sim (2,3,2,2)$.

Conversely, if $(y_1, ..., y_n) \in \langle(2, 3, ..., 2, 3, 2, 2)\rangle$ then we have

$$(y_1 - 2)3^{n-1} + (y_2 - 3)3^{n-2} + ... + (y_{n-1} - 2)3 + y_n - 2 = 0. \tag{9}$$

Hence, $y_n = 2$ or $y_n = 5$.
a) If $y_n = 2$ then $y_n - 2 = 0$. Hence, from (9), we get

$$(y_1, ..., y_{n-1}) \in \langle(2, 3, ..., 2, 3, 2)\rangle = \langle(x_1, ..., x_{n-1})\rangle.$$

Therefore, $(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-1}, 2)\rangle$.
b) If $y_n = 5$ then $y_n - 2 = 3$. Hence, from (9), we get

$$(y_1 - 2)3^{n-2} + (y_2 - 3)3^{n-3} + ... + (y_{n-2} - 3)3 + y_{n-1} - 1 = 0. \tag{10}$$

This implies $y_{n-1} = 1$ or $y_{n-1} = 4$.

b1) If $y_{n-1} = 1$ then from (10) we have

$$(y_1, ..., y_{n-2}) \in \langle(2, 3, ..., 2, 3)\rangle = \langle(x_1, ..., x_{n-2})\rangle.$$

Therefore, $(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-2}, 1, 5)\rangle$.

b2) If $y_{n-1} = 4$ then from (10) we have

$$(y_1, ..., y_{n-2}) \in \langle(2, 3, ..., 2, 3, 2, 2)\rangle = \langle(x'_1, ..., x'_{n-2})\rangle.$$

Therefore, $(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-2}, 4, 5)\rangle$.
ii) The case n is odd.
a) Clearly that if $(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-1}, 3)\rangle$ then

$$(y_1, ..., y_n) \in \langle(2, 3, ..., 2, 3, 3)\rangle = \langle(x'_1, ..., x'_n)\rangle.$$

b) If $(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-2}, 4, 0)\rangle = \langle(2, 3, ..., 2, 3, 2, 4, 0)\rangle$ then by (1) we have

$$(y_1, ..., y_n) \in \langle(2, 3, ..., 2, 3, 3)\rangle = \langle(x'_1, ..., x'_n)\rangle,$$

since $(4, 0) \sim (3, 3)$.
c) If $(y_1, ..., y_n) \in \langle(x'_1, ..., x'_{n-2}, 1, 0)\rangle = \langle(2, 3, ..., 2, 3, 3, 1, 0)\rangle$ then by (1) we have

$$(y_1, ..., y_n) \in \langle(2, 3, ..., 2, 3, 3)\rangle = \langle(x'_1, ..., x'_n)\rangle,$$

since $(3, 1, 0) \sim (2, 3, 3)$.

Conversely, if $(y_1, ..., y_n) \in \langle(x'_1, ..., x'_n)\rangle = \langle(2, 3, ..., 2, 3, 3)\rangle$, then we have

$$(y_1 - 2)3^{n-1} + (y_2 - 3)3^{n-2} + ... + (y_{n-1} - 3)3 + y_n - 3 = 0. \tag{11}$$

Hence, $y_n = 3$ or $y_n = 0$.
a) If $y_n = 3$ then $y_n - 3 = 0$. Hence, from (11) we have

$$(y_1, ..., y_{n-1}) \in \langle(2, 3, ..., 2, 3)\rangle = \langle(x_1, ..., x_{n-1})\rangle.$$

Therefore, by (1) we have $(y_1, ..., y_n) \in \langle(x_1, ..., x_{n-1}, 3)\rangle$.
b) If $y_n = 0$ then $y_n - 3 = -1$. Hence, from (11) we have

$$(y_1 - 2)3^{n-2} + (y_2 - 3)3^{n-3} + ... + (y_{n-2} - 2)3 + y_{n-1} - 4 = 0. \tag{12}$$

This implies $y_{n-1} = 1$ or $y_{n-1} = 4$.

b1) If $y_{n-1} = 1$ then $y_{n-1} - 1 = -3$. Hence, from (12) we have

$$(y_1, ..., y_{n-2}) \in \langle(2, 3, ..., 2, 3, 3)\rangle = \langle(x'_1, ..., x'_{n-2})\rangle.$$

This implies $(y_1, ..., y_n) \in \langle(x'_1, ..., x'_{n-2}, 1, 0)\rangle$.

b2) If $y_{n-1} = 4$ then $y_{n-1} - 4 = 0$. Hence, from (12) we have

$$(y_1, ..., y_{n-2}) \in \langle (2, 3, ..., 2, 3, 2) \rangle = \langle (x_1, ..., x_{n-2}) \rangle.$$

This implies $(y_1, ..., y_n) \in \langle (x_1, ..., x_{n-2}, 4, 0) \rangle$. The proposition is proved. $\square$

From Proposition 4, we have the following corollary, which will be used to establish the recurrence formula of $\mu_n(s_n)$ for each $n \in \mathbb{N}$.

Corrolary 2. *Let* $x = (x_1, x_2, ...) = (2, 3, 2, 3, ...) \in D^\infty$*. For each* $n \in \mathbb{N}$*, put* $s_n = \sum_{i=1}^{n} 3^{-i} x_i$ *and* $s'_n = \sum_{i=1}^{n} 3^{-i} x'_i$*, where* $(x'_1, ..., x'_{n-1}, x'_n) = (x_1, ..., x_{n-1}, x_{n-1})$*, we have*

$$\mu_n(s'_n) = \frac{10}{2^5}\mu_{n-1}(s_{n-1}) + \frac{5}{2^{10}}\big(\mu_{n-2}(s_{n-2}) + \mu_{n-2}(s'_{n-2})\big).$$

Proof. By Proposition 4, we have
a) If n is even then

$$\langle (x'_1, ..., x'_n) \rangle = \langle (x_1, ..., x_{n-1}, 2) \rangle \cup \langle (x_1, ..., x_{n-2}, 1, 5) \rangle \cup \langle (x'_1, ..., x'_{n-2}, 4, 5) \rangle.$$

Therefore,

$$\begin{aligned}\mu_n(s'_n) &= \frac{10}{2^5}\mu_{n-1}(s_{n-1}) + \frac{1}{2^5}\cdot\frac{5}{2^5}\mu_{n-2}(s_{n-2}) + \frac{1}{2^5}\cdot\frac{5}{2^5}\mu_{n-2}(s'_{n-2}) \\ &= \frac{10}{2^5}\mu_{n-1}(s_{n-1}) + \frac{5}{2^{10}}[\mu_{n-2}(s_{n-2}) + \mu_{n-2}(s'_{n-2})].\end{aligned}$$

b) If n is odd then

$$\langle (x'_1, ..., x'_n) \rangle = \langle (x_1, ..., x_{n-1}, 3) \rangle \cup \langle (x_1, ..., x_{n-2}, 4, 0) \rangle \cup \langle (x'_1, ..., x'_{n-2}, 1, 0) \rangle.$$

Therefore,

$$\begin{aligned}\mu_n(s'_n) &= \frac{10}{2^5}\mu_{n-1}(s_{n-1}) + \frac{1}{2^5}\cdot\frac{5}{2^5}\mu_{n-2}(s_{n-2}) + \frac{1}{2^5}\cdot\frac{5}{2^5}\mu_{n-2}(s'_{n-2}) \\ &= \frac{10}{2^5}\mu_{n-1}(s_{n-1}) + \frac{5}{2^{10}}[\mu_{n-2}(s_{n-2}) + \mu_{n-2}(s'_{n-2})].\end{aligned}$$

Hence,

$$\mu_n(s'_n) = \frac{10}{2^5}\mu_{n-1}(s_{n-1}) + \frac{5}{2^{10}}\big(\mu_{n-2}(s_{n-2}) + \mu_{n-2}(s'_{n-2})\big).$$

The corollary is proved.

From Corollaries 1 and 2, we have

Corrolary 3. *Let* $x = (x_1, x_2, ...) = (2, 3, 2, 3, ...) \in D^\infty$*. For each* $n \in \mathbb{N}$*, put* $s_n = \sum_{i=1}^{n} 3^{-i} x_i$*. Then we have*

$$\mu_n(s_n) = \frac{10}{2^5}\mu_{n-1}(s_{n-1}) + \frac{15}{2^{10}}\mu_{n-2}(s_{n-2}) - \frac{45}{2^{15}}\mu_{n-3}(s_{n-3}).$$

Proof. By Corollaries 1 and 2, we have

$$\mu_n(s_n) = \frac{10}{2^5}\mu_{n-1}(s_{n-1}) + \frac{1}{2^5}\mu_{n-1}(s'_{n-1}) \tag{13}$$

$$\mu_{n-1}(s'_{n-1}) = \frac{10}{2^5}\mu_{n-2}(s_{n-2}) + \frac{5}{2^{10}}\big(\mu_{n-3}(s_{n-3}) + \mu_{n-3}(s'_{n-3})\big) \tag{14}$$

$$\mu_{n-2}(s_{n-2}) = \frac{10}{2^5}\mu_{n-3}(s_{n-3}) + \frac{1}{2^5}\mu_{n-3}(s'_{n-3}). \tag{15}$$

From (13), (14) and (15), the assertion of the corollary follows.

2.2 The proof of the main theorem

Lemma 2. *Let $x = (x_1, x_2, ...) = (2, 3, 2, 3, ...) \in D^\infty$. For each $n \in \mathbb{N}$, put $s_n = \sum_{i=1}^{n} 3^{-i} x_i$. Then we have $\mu_n(s_n) \geq \mu_n(t_n)$ for all $t_n \in$ supp μ_n.*

We will prove the lemma by induction. For $n = 1$ we have

$$\mu_1(s_1) = P(X_1 = 2) = \frac{10}{2^5} \geq \mu_1(t_1) \in \{\frac{1}{2^5},\ \frac{5}{2^5},\ \frac{10}{2^5}\}$$

for all $t_1 \in$ supp μ_1. Assume that the lemma is true for $n = k$, i.e.,

$$\mu_k(s_k) \geq \mu_k(t_k) \text{ for all } t_k \in \text{ supp } \mu_k.$$

We will show that the lemma is true for $n = k+1$. For any $y = (y_1, y_2, ...) \in D^\infty$, put $t_n = \sum_{i=1}^{n} 3^{-i} y_i$ for each $n \in \mathbb{N}$, then $t_{k+1} = \sum_{i=1}^{k+1} 3^{-i} y_i$. We consider the following cases of y_{k+1}.

Case 1. If $y_{k+1} = 1$ (or 4), then by Lemma 1, t_{k+1} has at most two representations

$$t_{k+1} = t_k + 1.3^{-(k+1)} = t'_k + 4.3^{-(k+1)}.$$

Therefore, by induction hypothesis, we have

$$\mu_{k+1}(t_{k+1}) = \mu_k(t_k)P(X_{k+1} = 1) + \mu_k(t'_k)P(X_{k+1} = 4)$$

$$\leqslant \mu_k(t_k)(\frac{5}{2^5} + \frac{5}{2^5}) = \frac{10}{2^5}\mu_k(t_k).$$

By Corollary 1 (ii), we have

$$\mu_{k+1}(s_{k+1}) > \frac{10}{2^5}\mu_k(s_k) \geq \mu_{k+1}(t_{k+1}).$$

Case 2. If $y_{k+1} = 0$ (or 3), then by Lemma 1, t_{k+1} has at most two representations

$$t_{k+1} = t_k + 0.3^{-(k+1)} = t'_k + 3.3^{-(k+1)}.$$

a) If $y_k = 0$ (or 3), then $(y_k, y_{k+1}) \in \{(0, 0),\ (0, 3)\}$. Therefore, by Lemma 1 we have $(y'_1, ..., y'_{k+1}) \in \langle(y_1, ..., y_{k+1})\rangle$ iff

$$(y'_k, y'_{k+1}) \in \{(0, 0),\ (3, 0),\ (2, 3),\ (5, 3),\ (0, 3),\ (1, 0),\ (3, 3),\ (4, 0), \}.$$

By induction hypothesis, we have

$$\begin{aligned}\mu_{k+1}(t_{k+1}) \leqslant\ & \mu_{k-1}(s_{k-1})[P(X_k=0)P(X_{k+1}=0)+P(X_k=3)P(X_{k+1}=0)\\ &+P(X_k=2)P(X_{k+1}=3)+P(X_k=5)P(X_{k+1}=3)\\ &+P(X_k=0)P(X_{k+1}=3)+P(X_k=3)P(X_{k+1}=3)\\ &+P(X_k=1)P(X_{k+1}=0)+P(X_k=4)P(X_{k+1}=0)]\\ &=\mu_{k-1}(s_{k-1})(\frac{1}{2^5}.\frac{1}{2^5}+\frac{10}{2^5}.\frac{1}{2^5}+\frac{10}{2^5}.\frac{1}{2^5}+\frac{10}{2^5}.\frac{10}{2^5}\\ &+\frac{1}{2^5}.\frac{10}{2^5}+\frac{10}{2^5}.\frac{10}{2^5}+\frac{5}{2^5}.\frac{1}{2^5}+\frac{5}{2^5}.\frac{1}{2^5})\\ &=\frac{241}{2^{10}}\mu_{k-1}(s_{k-1}).\end{aligned}$$

By hypothesis induction and Corollary 1 (ii), we have

$$\mu_{k+1}(s_{k+1})>\frac{10}{2^5}\mu_k(s_k)\geq\frac{241}{2^{10}}\mu_{k-1}(s_{k-1})=\mu_{k+1}(t_{k+1}).$$

b) If $y_k=4$ (or 1), then $(y_k,y_{k+1})\in\{(4,0),\ (4,3)\}.$ Therefore, by Lemma 1 we have $(y'_1,...,y'_{k+1})\in\langle(y_1,...,y_{k+1})\rangle$ iff

$$(y'_k,y'_{k+1})\in\{(2,0),\ (5,0),\ (1,3),\ (4,3),\ (0,3),\ (1,0),\ (3,3),\ (4,0),\}.$$

By induction hypothesis, we have

$$\begin{aligned}\mu_{k+1}(t_{k+1}) \leqslant\ & \mu_{k-1}(t_{k-1})[P(X_k=2)P(X_{k+1}=0)+P(X_k=5)P(X_{k+1}=0)\\ &+P(X_k=1)P(X_{k+1}=3)+P(X_k=4)P(X_{k+1}=3)\\ &+P(X_k=0)P(X_{k+1}=3)+P(X_k=1)P(X_{k+1}=0)\\ &+P(X_k=3)P(X_{k+1}=3)+P(X_k=4)P(X_{k+1}=0)]\\ &=\mu_{k-1}(s_{k-1})(\frac{10}{2^5}.\frac{1}{2^5}+\frac{1}{2^5}.\frac{1}{2^5}+\frac{5}{2^5}.\frac{10}{2^5}+\frac{5}{2^5}.\frac{10}{2^5}\\ &+\frac{1}{2^5}.\frac{10}{2^5}+\frac{5}{2^5}.\frac{1}{2^5}+\frac{10}{2^5}.\frac{10}{2^5}+\frac{5}{2^5}.\frac{1}{2^5})\\ &=\frac{231}{2^{10}}\mu_{k-1}(s_{k-1}).\end{aligned}$$

By hypothesis induction and Corollary 1 (ii), we have

$$\mu_{k+1}(s_{k+1})>\frac{10}{2^5}\mu_k(s_k)\geq\frac{231}{2^{10}}\mu_{k-1}(s_{k-1})\geq\mu_{k+1}(t_{k+1}).$$

c) If $y_k=2$ (or 5), then $(y_k,y_{k+1})\in\{(2,0),\ (2,3)\}.$ Therefore, by Lemma 1 we have $(y'_1,...,y'_{k+1})\in\langle(y_1,...,y_{k+1})\rangle$ iff

$$(y'_k,y'_{k+1})\in\{(0,0),\ (3,0),\ (2,3),\ (5,3),\ (1,3),\ (4,3),\ (2,0),\ (5,0),\}.$$

By induction hypothesis, we have

$$\begin{aligned}
\mu_{k+1}(t_{k+1}) \leqslant\ & \mu_{k-1}(t_{k-1})[P(X_k=0)P(X_{k+1}=0)+P(X_k=3)P(X_{k+1}=0)\\
&+P(X_k=2)P(X_{k+1}=3)+P(X_k=5)P(X_{k+1}=3)\\
&+P(X_k=2)P(X_{k+1}=0)+P(X_k=5)P(X_{k+1}=0)\\
&+P(X_k=1)P(X_{k+1}=3)+P(X_k=4)P(X_{k+1}=3)]\\
&=\mu_{k-1}(s_{k-1})(\frac{1}{2^5}.\frac{1}{2^5}+\frac{10}{2^5}.\frac{1}{2^5}+\frac{10}{2^5}.\frac{1}{2^5}+\frac{10}{2^5}.\frac{10}{2^5}\\
&+\frac{5}{2^5}.\frac{10}{2^5}+\frac{5}{2^5}.\frac{10}{2^5}+\frac{10}{2^5}.\frac{1}{2^5}+\frac{1}{2^5}.\frac{1}{2^5})\\
&=\frac{231}{2^{10}}\mu_{k-1}(s_{k-1}).
\end{aligned}$$

Therefore, by Corollary 1 (ii), we have

$$\mu_{k+1}(s_{k+1})>\frac{10}{2^5}\mu_k(s_k)\geq\frac{231}{2^{10}}\mu_{k-1}(s_{k-1})\geq\mu_{k+1}(t_{k+1}).$$

Case 3. If $y_{k+1}=2$ (or 5). This case is proved similarly to the Case 2.

Therefore, the lemma is proved.

By resolving Fibonacci recurrence formula of $\mu_n(s_n)$ in Corollary 3, we have the following corollary.

Corrolary 4. *Let $x=(x_1,x_2,...)=(2,3,2,3,...)\in D^\infty$. For each $n\in\mathbb{N}$, put $s_n=\sum_{i=1}^{n}3^{-i}x_i$. Then we have*

$$\mu_n(s_n)=a_1X_1^n+a_2X_2^n+a_3X_3^n$$

for

$$X_1=\frac{2}{3.2^5}[\sqrt{145}\cos(\frac{\arccos\frac{427}{59\sqrt{145}}}{3})+5]\simeq 0,3435055158$$

$$X_2=\frac{-2}{3.2^5}[\sqrt{145}\cos(\frac{\arccos\frac{427}{59\sqrt{145}}}{3}+\frac{\pi}{3})+5]\simeq 0,04959875748$$

$$X_3=\frac{-2}{3.2^5}[\sqrt{145}\cos(\frac{\arccos\frac{427}{59\sqrt{145}}}{3}-\frac{\pi}{3})+5]\simeq -0,08060427328$$

and $a_1,\ a_2,\ a_3$ are roots of the following system of three equations

$$\begin{aligned}
\mu_1(s_1)&=a_1X_1+a_2X_2+a_3X_3\\
\mu_2(s_2)&=a_1X_1^2+a_2X_2^2+a_3X_3^2\\
\mu_3(s_3)&=a_1X_1^3+a_2X_2^3+a_3X_3^3,
\end{aligned}$$

where $\mu_1(s_1),\ \mu_2(s_2),\ \mu_3(s_3)$ are the values in Corollary 1.

From Lemma 2, Corollary 3 and Proposition 2, we have

Theorem. *Let μ is the $5-$fold convolution of the standard Cantor measure, then the lower extreme value of the local dimension of μ is*

$$\underline{\alpha}_5=|\frac{\log\left[\frac{2}{3.2^5}\left(\sqrt{145}\cos(\frac{\arccos\frac{427}{59\sqrt{145}}}{3})+5\right)\right]}{\log 3}|\simeq 0,972638.$$

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