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On the Oscillation, the Convergence, and the Boundedness of Solutions for a Neutral Difference Equation

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Abstract. In this paper, the oscillation, convergence and boundedness for neutral difference equations

$$\Delta(x_n + \delta_n x_{n-\tau}) + \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) = 0, \quad n = 0, 1, \cdots$$

are investigated.

Keywork: Neutral difference equation, oscillation, nonoscillation, convergence, boundedness.

1. Introduction

Recently there has been a considerable interest in the oscillation of the solutions of difference equations of the form

$$\Delta(x_n + \delta x_{n-\tau}) + \alpha(n) x_{n-\sigma} = 0,$$

where $n \in \mathbb{N}$, the operator Δ is defined as $\Delta x_n = x_{n+1} - x_n$, the function $\alpha(n)$ is defined on $\mathbb{N}$, δ is a constant, τ is a positive integer and σ is a nonnegative integer, (see for example the work in [1-7] and the references cited therein).

In [2], the author obtained some sufficient criterions for the oscillation and convergence of solutions of the difference equation

$$\Delta(x_n + \delta x_{n-\tau}) + \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) = 0,$$

for $n \in \mathbb{N}, n \geqslant a$ for some $a \in \mathbb{N}$, the operator Δ is defined as $\Delta x_n = x_{n+1} - x_n$, δ is a constant, $\tau, r, m_1, m_2, \cdots, m_r$ are fixed positive integers, and the functions $\alpha_i(n)$ are defined on $\mathbb{N}$ and the function F is defined on $\mathbb{R}$.

Motivated by the work above, in this paper, we aim to study the oscillation and asymptotic behavior for neutral difference equation

$$\Delta(x_n + \delta_n x_{n-\tau}) + \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) = 0, \tag{1}$$

where $\delta_n, \quad n \in \mathbb{N}$ is not zero for infinitely many values of n and $F : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

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Put $A = \max\{\tau, m_1, \cdots, m_r\}$. Then, by a solution of (1) we mean a function which is defined for $n \geqslant -A$ and sastisfies the equation (1) for $n \in \mathbb{N}$. Clearly, if

$$x_n = a_n, \quad n = -A, -A+1, \cdots, -1, 0$$

are given, then (1) has a unique solution, and it can be constructed recursively.

A nontrivial solution $\{x_n\}_{n \ n_0}$ of (1) is called oscillatory if for any $n_1 \geqslant n_0$ there exists $n_2 \geqslant n_1$ such that $x_{n_2}x_{n_2+1} \leqslant 0$. The difference equation (1) is called oscillatory if all its solutions are oscillatory. If the solution $\{x_n\}_{n \ n_0}$ is not oscillatory then it is said to be nonoscillatory. Equivalently, the solution $\{x_n\}_{n \ n_0}$ is nonoscillatory if it is eventually positive or negative, i.e. there exists an integer $n_1 \geqslant n_0$ such that $x_n x_{n+1} > 0$ for all $n \geqslant n_1$.

2. Main results

To begin with, we assume that

$$xF(x) > 0 \text{ for } x \neq 0. \tag{2}$$

By an argument analogous to that used for the proof of Lemma 3, Theorem 6 and Theorem 7 in [2], we get the following results.

Lemma 1. *Let $\{x_n\}$ be a nonoscillatory solution of (1). Put $z_n = x_n + \delta_n x_{n-\tau}$.*

(i) If $\{x_n\}$ is eventually positive (negative), then $\{z_n\}$ is eventually nonincreasing (nondecreasing).

(ii) If $\{x_n\}$ is eventually positive (negative) and there exists a constant γ such that

$$-1 < \gamma \leqslant \delta_n, \quad \forall n \in \mathbb{N}$$

then eventually $z_n > 0$ ($z_n < 0$).

Theorem 1. *Suppose there exist positive constants $\alpha_i (i = 1, 2, \cdots, r)$ and M such that*

$$\alpha_i(n) \geqslant \alpha_i, \quad \forall n \in \mathbb{N},$$

$$|F(x)| \geqslant M|x|, \quad \forall x,$$

$$\delta_n \geqslant 0, \quad \forall n \in \mathbb{N}.$$

Then, every nonoscillatory solution of (1) tend to 0 as $n \to \infty$.

Theorem 2. *Assume that*

$$\sum_{\ell=1}^{\infty} \sum_{i=1}^{r} \alpha_i(\ell) = \infty, \tag{3}$$

and there exists a constant η such that

$$-1 < \eta \leqslant \delta_n \leqslant 0, \quad \forall n \in \mathbb{N}. \tag{4}$$

Suppose further that, if $|x| \geqslant c$ then $|F(x)| \geqslant c_1$ where c and c_1 are positive constants. Then, every nonoscillatory solution of (1) tends to 0 as $n \to \infty$.

Theorem 3. *Assume that the given hypothese in Theorem 2 are satisfied. If F is a nondecreasing function such that*

$$\int_0^{\alpha} \frac{dt}{F(t)} < \infty \text{ and } \int_{-\alpha}^{0} \frac{dt}{F(t)} > -\infty \quad \text{for all } \alpha > 0, \tag{5}$$

then the equation (1) is oscillatory.

Proof. Suppose that (1) has a nonoscillatoty solution $\{x_n\}$. If $x_n > 0$ for $n \geqslant n_0$, then by Lemma 1 there exists a $n_1 \geqslant n_0$ such that $x_{n-\tau} > 0, x_{n-m_i} > 0 \ (1 \leqslant i \leqslant r), z_n > 0$ and $\Delta z_n \leqslant 0$ for $n \geqslant n_1$. Put $z_n = x_n + \delta_n x_{n-\tau}$ and $m_* = \max_{1 \leqslant i \leqslant r} m_i$. We note that (4) implies that $z_n \leqslant x_n$ and from (1), we have

$$\Delta z_n + \sum_{i=1}^{r} \alpha_i(n) F(z_{n-m_i}) \leqslant 0$$

and so

$$\Delta z_n + \sum_{i=1}^{r} \alpha_i(n) F(z_n) \leqslant 0 \quad \text{for } n \geqslant n_2 = n_1 + m_*$$

or

$$\sum_{i=1}^{r} \alpha_i(n) \leqslant -\frac{\Delta z_n}{F(z_n)} \quad \text{for } n \geqslant n_2 = n_1 + m_*.$$

Now for $z_{n+1} \leqslant t \leqslant z_n$ we have $F(t) \leqslant F(z_n)$, and so

$$\sum_{i=1}^{r} \alpha_i(n) \leqslant \int_{z_{n+1}}^{z_n} \frac{dt}{F(t)} \quad \text{for } n \geqslant n_2.$$

Summing both sides of the above inequality from n_2 to n and taking the limit as $n \to \infty$, we get

$$\sum_{\ell=n_2}^{\infty} \sum_{i=1}^{r} \alpha_i(\ell) \leqslant \int_{z_{n+1}}^{z_{n_2}} \frac{dt}{F(t)} < \int_0^{z_{n_2}} \frac{dt}{F(t)} < \infty,$$

which contradicts (3). The proof for the case $\{x_n\}$ eventually negative is similar.

Example 1. Consider the difference equation

$$\Delta\Big(x_n + \frac{1-n}{2n} x_{n-2}\Big) + \sum_{i=1}^{2} \frac{1}{n+i} x_{n-i}^{\frac{1}{3}} = 0, \quad n \geqslant 1. \tag{6}$$

It is clear that this equation is a particular case of (1), where $\delta_n = \frac{1-n}{2n}$, $\alpha_i(n) = \frac{1}{n+i}, \forall n \in \mathbb{N}, i = 1, i = 2$ and $F(x) \equiv x^{\frac{1}{3}}$.

It is easy to verify that all conditions of Theorem 3 hold. Hence, the equation (6) is oscillatory.

Theorem 4. *Assume that the first and the third condition in Theorem 2 are satisfied and there exists constants σ, μ such that*

$$\mu \leqslant \delta_n \leqslant \sigma < -1. \tag{7}$$

Suppose further that, $\tau > m_* = \max_{1 \leqslant i \leqslant r} m_i$ and F is a nondecreasing function such that

$$\int_{\epsilon}^{\infty} \frac{dt}{F(t)} < \infty \text{ and } \int_{-\infty}^{-\epsilon} \frac{dt}{F(t)} < \infty \quad \text{for all } \epsilon > 0, \tag{8}$$

then the equation (1) is oscillatory.

Proof. Suppose that (1) has a nonoscillatoty solution $\{x_n\}$, $x_n > 0$ for $n \geqslant n_0$. From Lemma 1 there exists a $n_1 \geqslant n_0$ such that $x_{n-\tau} > 0, x_{n-m_i} > 0 \ (1 \leqslant i \leqslant r), z_n < 0$ and $\Delta z_n \leqslant 0$ for $n \geqslant n_1$. Then from (7) we have

$$\mu x_{n-\tau} \leqslant \delta_n x_{n-\tau} < z_n < 0$$

and hence

$$0 < \frac{z_{n+\tau}}{\mu} < 0, \quad \text{for } n \geqslant n_1.$$

Thus, it follows that

$$F\Big(\frac{z_{n+\tau-m_i}}{\mu}\Big) \leqslant F(x_{n-m_i}) \quad \text{for } n \geqslant n_2 \geqslant n_1 + m_*, 1 \leqslant i \leqslant r.$$

Since $n + \tau - m_i \geqslant n + 1, 1 \leqslant i \leqslant r$ the above inequality gives

$$F\Big(\frac{z_{n+1}}{\mu}\Big) \leqslant F\Big(\frac{z_{n+\tau-m_i}}{\mu}\Big) \leqslant F(x_{n-m_i}), \quad 1 \leqslant i \leqslant r.$$

Hence, from (1) we find

$$\Delta z_n + \sum_{i=1}^{r} \alpha_i(n) F\Big(\frac{z_{n+1}}{\mu}\Big) \leqslant 0$$

or

$$\sum_{i=1}^{r} \alpha_i(n) \leqslant -\frac{\Delta z_n}{F\Big(\frac{z_{n+1}}{\mu}\Big)} \quad \text{for } n \geqslant n_2. \tag{9}$$

Now for $\frac{z_n}{\mu} \leqslant t \leqslant \frac{z_{n+1}}{\mu}$ we have $F\Big(\frac{z_{n+1}}{\mu}\Big) \geqslant F(t)$, and so

$$\frac{1}{\mu} \frac{\Delta z_n}{F\Big(\frac{z_{n+1}}{\mu}\Big)} \leqslant \int_{\frac{z_n}{\mu}}^{\frac{z_{n+1}}{\mu}} \frac{dt}{F(t)} \quad \text{for } n \geqslant n_2. \tag{10}$$

Using (10) in (9) and summing both sides from n_2 to n and taking the limit as $n \to \infty$, we get

$$\sum_{\ell=n_2}^{\infty} \sum_{i=1}^{r} \alpha_i(\ell) \leqslant -\mu \int_{\frac{z_{n_2}}{\mu}}^{\frac{z_{n+1}}{\mu}} \frac{dt}{F(t)} \quad \text{for } n \geqslant n_2.$$

But this in view of (8) contradicts (7). The proof for the case $\{x_n\}$ eventually negative is similar.

Example 2. Consider the difference equation

$$\Delta\Big(x_n - \frac{1+2n}{n} x_{n-2}\Big) + \sum_{i=1}^{2} \frac{i}{n+i} x_{n-i}^3 = 0, \quad n \geqslant 1. \tag{11}$$

It is clear that this equation is a particular case of (1), where $\delta_n = -\frac{1+2n}{n}$, $\alpha_i(n) = \frac{i}{n+i}, \forall n \in \mathbb{N}, i = 1, i = 2$ and $F(x) \equiv x^3$.

It can be verified that all conditions of Theorem 4 hold. Hence, the equation (11) is oscillatory.

Theorem 5. *Suppose that* $\delta_n \geqslant 0, \quad n \in \mathbb{N}$. *Then, all unbounded solutions of the equation (1) are oscillatory.*

Proof. Suppose the contrary. Without loss of generality, let $\{x_n\}$ be an unbounded and eventually positive solution of (1). By Lemma 1, we have $z_n > 0$ and $\Delta z_n \leqslant 0$ eventually. Hence, there exists $\lim_{n\to\infty} z_n$. Put $\lim_{n\to\infty} z_n = \beta$. We have

$$\beta \in [0, \infty). \tag{12}$$

Now, in view of $\delta_n \geqslant 0, \quad n \in \mathbb{N}$ we have $z_n \geqslant x_n$ and (12) show that $\{x_n\}$ is bounded, which is a contradiction.

From now we always assume that

$$xF(x) < 0 \text{ for } x \neq 0. \tag{13}$$

Theorem 6. *Assume that* $\delta_n \geqslant 0, \quad n \in \mathbb{N}, \sum_{\ell=1}^{\infty}\sum_{i=1}^{r} \alpha_i(\ell) < \infty$ *and* F *is nonincreasing. Suppose further that*

$$\int_c^{\infty} \frac{dt}{F(t)} = -\infty \text{ and } \int_{-\infty}^{-c} \frac{dt}{F(t)} = \infty \quad \text{for all } c > 0. \tag{14}$$

Then, all nonoscillatory solutions of the equation (1) are bounded.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of (1), and let $n_0 \in \mathbb{N}$ be such that $|x_n| \neq 0$ for all $n \geqslant n_0$. Assume that $x_n > 0$ for all $n \geqslant n_0$. Put $m_* = \max_{1\ i\ r}$ and $n_1 = n_0 + \tau + m_*$. We have $x_{n-\tau-m_i} > 0$ for all $n \geqslant n_1$ and $1 \leqslant i \leqslant r$. Put $z_n = x_n + \delta_n x_{n-\tau}$. We have $z_n > 0$ and $\Delta z_n = -\sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) \geqslant 0$ for all $n \geqslant n_1$. Hence, $\{z_n\}$ is nondecreasing and satisfies $z_n \geqslant x_n$ for all $n \geqslant n_1$. Therefore, we find

$$\begin{aligned}\Delta z_n = -\sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) &\leqslant -\sum_{i=1}^{r} \alpha_i(n) F(z_{n-m_i}) \\ &\leqslant -\sum_{i=1}^{r} \alpha_i(n) F(z_n),\end{aligned}$$

or

$$-\frac{\Delta z_n}{F(z_n)} \leqslant \sum_{i=1}^{r} \alpha_i(n), \quad \forall n \geqslant n_1. \tag{15}$$

Since $t \in [z_n, z_{n+1}]$, $F(t) \leqslant F(z_n)$. By (15) we obtain

$$-\int_{z_n}^{z_{n+1}} \frac{dt}{F(t)} \leqslant -\frac{\Delta z_n}{F(z_n)} \leqslant \sum_{i=1}^{r} \alpha_i(n), \quad \forall n \geqslant n_1. \tag{16}$$

Summing the inequality (16) from n_1 to $n-1$ and taking the limit as $n \to \infty$, we have

$$-\int_{z_{n_1}}^{z_n} \frac{dt}{F(t)} \leqslant \sum_{\ell=n_1}^{n-1} \sum_{i=1}^{r} \alpha_i(\ell). \tag{17}$$

From (17) and the hypothese of Theorem 6 we find that $\{z_n\}$ is bounded from above. Since $0 < x_n \leqslant z_n$, $\{x_n\}$ is also bounded from above. The proof is similar when $\{x_n\}$ is eventually negative.

Example 3. Consider the difference equation

$$\Delta\Big(x_n + 2^n x_{n-2}\Big) + \sum_{i=1}^{2} \frac{1}{(i+1)^n}(-x_{n-i}^{\frac{1}{3}}) = 0, \quad n \geqslant 1. \tag{18}$$

It is clear that this equation is a particular case of (1), where $\delta_n = 2^n$, $\alpha_i(n) = \frac{1}{(i+1)^n}, \forall n \in \mathbb{N}, i = 1, i = 2$ and $F(x) \equiv -x^{\frac{1}{3}}$.

It can be verified that all conditions of Theorem 6 hold. Hence, all nonoscillatory solutions of the equation (18) are bounded.

Corollary. *Suppose that the assumptions of Theorem 6 hold. Further, suppose that* $\{\delta_n\}$ *tends to* 0 *as* $n \to \infty$. *Then, every nonoscillatory solution of (1) tends to* 0 *as* $n \to \infty$.

Proof. Let $\{x_n\}$ be an eventually positive solution of (1). By Theorem 6, $\{z_n\}$ is eventually positive, nondecreasing and bounded above. Thus, there exists a constant $C > 0$ such that

$$\delta_n x_{n-\tau} < z_n < C$$

for sufficiently large n. Hence,

$$x_{n-\tau} < \frac{C}{\delta_n} \to 0 \text{ as } n \to \infty.$$

Theorem 7. *Assume that*

$$\sum_{\ell=1}^{\infty} \sum_{i=1}^{r} \alpha_i(\ell) = \infty, \tag{19}$$

and there exists a constant $\delta > 0$ *such that*

$$\delta_n \leqslant \delta, \quad \forall n \in \mathbb{N}. \tag{20}$$

Suppose further that, if $|x| \geqslant c$ *then* $|F(x)| \geqslant c_1$ *where* c *and* c_1 *are positive constants. Then, for every bounded nonoscillatory solution* $\{x_n\}$ *of (1) we have*

$$\liminf_{n\to\infty} |x_n| = 0.$$

Proof. Assume that, $\{x_n\}$ is a bounded nonoscillatory solution of (1). Then, there exists constants $c, C > 0$ such that $c \leqslant x_n \leqslant C$ for all $n \geqslant n_0 \in \mathbb{N}$. It implies that

$$z_n \leqslant (1 + \delta)C. \tag{21}$$

Put $m_* = \max\limits_{1 \ i \ r}$ and $n_1 = n_0 + \tau + m_*$. We have $x_{n-\tau-m_i} \geqslant c$ for all $n \geqslant n_1$ and $1 \leqslant i \leqslant r$. By the hypothese of Theorem 7, there exists a constant $c_1 > 0$ such that $|F(x_{n-m_i}| \geqslant c_1$ for all $n \geqslant n_1$ and $1 \leqslant i \leqslant r$. Thus,

$$\Delta z_n = -\sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) \geqslant \sum_{i=1}^{r} \alpha_i(n) c_1, \quad \forall n \geqslant n_1. \tag{22}$$

Summing the inequality (22) from n_1 to $n-1$, we obtain

$$z_n = z_{n_1} + c_1 \sum_{\ell=n_1}^{n-1} \sum_{i=1}^{r} \alpha_i(\ell) \to \infty \text{ as } n \to \infty,$$

which contradicts (22). The proof is complete.

Example 4. Consider the difference equation

$$\Delta\Big(x_n + \frac{2n-1}{n}x_{n-1}\Big) + \sum_{i=1}^{2}\frac{1}{n+i}(-x_{n-i}^{\alpha}) = 0, \quad n \geqslant 1, \tag{23}$$

where α is an odd integer. It is clear that this equation is a particular case of (1), where $\delta_n = \frac{2n-1}{n}$, $\alpha_i(n) = \frac{1}{n+i}, \forall n \in \mathbb{N}, i = 1, i = 2$ and $F(x) \equiv -x^{\alpha}$.

It can be verified that all conditions of Theorem 7 hold.

Theorem 8. *Assume that the conditions (3), (7) hold and F is a nonincreasing function such that*

$$\int_0^{\alpha}\frac{dt}{F(t)} < \infty \text{ and } \int_{-\alpha}^{0}\frac{dt}{F(t)} > -\infty \quad \text{for all } \alpha > 0.$$

Further, suppose that $m_i \geqslant \tau, \quad \forall 1 \leqslant i \leqslant r$. Then, every nonoscillatory solution $\{x_n\}$ of (1) satisfies $|x_n| \to \infty$ as $n \to \infty$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of (1). Assume that $\{x_n\}$ is eventually positive. Then, there exists $n_0 \in \mathbb{N}$ such that $x_{n-\tau-m_i} > 0$ for all $n \geqslant n_0$ and $1 \leqslant i \leqslant r$. Put $z_n = x_n + \delta_n x_{n-\tau}$. Then, since $\Delta z_n = -\sum_{i=1}^{r}\alpha_i(n)F(x_{n-m_i}) \geqslant 0$ for all $n \geqslant n_0$, $\{z_n\}$ is nondecreasing for $n \geqslant n_0$. Therefore, $z_n \to L > -\infty$ as $n \to \infty$. If $L \leqslant 0$ then $z_n < 0$ for all $n \geqslant 0$ and hence

$$0 > z_n = x_n + \delta_n x_{n-\tau} > \eta x_{n-\tau}, \quad n \geqslant n_0.$$

It implies $z_{n+\tau} > \eta x_n, \quad n \geqslant n_0$ or $x_n > \frac{z_{n+\tau}}{\eta}, \quad n \geqslant n_0$. Now since $m_i \geqslant \tau, \quad \forall 1 \leqslant i \leqslant r$ and F is nonincreasing, we have

$$\Delta z_n \geqslant -\sum_{i=1}^{r}\alpha_i(n)F\Big(\frac{z_{n+\tau-m_i}}{\eta}\Big) \geqslant -\sum_{i=1}^{r}\alpha_i(n)F\Big(\frac{z_n}{\eta}\Big),$$

or

$$-\frac{\Delta z_n}{F\Big(\frac{z_n}{\eta}\Big)} \geqslant \sum_{i=1}^{r}\alpha_i(n).$$

Now for $\frac{z_{n+1}}{\eta} \leqslant t \leqslant \frac{z_n}{\eta}$ we have $-\frac{1}{F(t)} \geqslant -\frac{1}{F\Big(\frac{z_n}{\eta}\Big)}$, and so

$$-\int_{\frac{z_{n+1}}{\eta}}^{\frac{z_n}{\eta}}\frac{dt}{F(t)} \geqslant -\int_{\frac{z_{n+1}}{\eta}}^{\frac{z_n}{\eta}}\frac{1}{F\Big(\frac{z_n}{\eta}\Big)}\sum_{i=1}^{r}\alpha_i(n) = -\frac{\Delta z_n}{(-\eta)F\Big(\frac{z_n}{\eta}\Big)} \quad \text{for } n \geqslant n_0,$$

or

$$\eta\int_{\frac{z_{n+1}}{\eta}}^{\frac{z_n}{\eta}}\frac{dt}{F(t)} \geqslant -\frac{\Delta z_n}{F\Big(\frac{z_n}{\eta}\Big)} \geqslant \sum_{i=1}^{r}\alpha_i(n) \quad \text{for } n \geqslant n_0. \tag{24}$$

Summing both sides of the inequality (24) from n_0 to n and taking the limit as $n \to \infty$, we get

$$\eta\int_{\frac{L}{\eta}}^{\frac{z_{n_0}}{\eta}}\frac{dt}{F(t)} \geqslant \sum_{\ell=n_0}^{\infty}\sum_{i=1}^{r}\alpha_i(\ell),$$

which contradicts (3). Thus, $L > 0$. Now let $n_1 \geqslant n_0$ be such that $0 < z_n \leqslant x_n + \sigma x_{n-\tau}$ for $n \geqslant n_1$. Then, $x_n \geqslant -\sigma x_{n-\tau}$ and by induction, we have $x_{n+j\tau} \geqslant (-\sigma)^j x_{n-\tau}$ for each positive integer j. This implies that $x_n \to \infty$ as $n \to \infty$. The proof is similar when $\{x_n\}$ is eventually negative.

Example 5. Consider the difference equation

$$\Delta\Big(x_n - \frac{2+3n}{2n}x_{n-1}\Big) + \sum_{i=1}^{2} \frac{1}{n+i}(-x_{n-i}^{\frac{1}{3}}) = 0, \quad n \geqslant 1. \tag{25}$$

It is clear that this equation is a particular case of (1), where $\delta_n = -\frac{2+3n}{2n}$, $\alpha_i(n) = \frac{1}{n+i}, \forall n \in \mathbb{N}, i = 1, i = 2$ and $F(x) \equiv -x^{\frac{1}{3}}$.

It can be verified that all conditions of Theorem 8 hold.

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