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Stability Radii for Difference Equations with Time-varying Coefficients

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Abstract. This paper deals with a formula of stability radii for an linear difference equation (LDEs for short) with the coefficients varying in time under structured parameter perturbations. It is shown that the l_p- real and complex stability radii of these systems coincide and they are given by a formula of input-output operator. The result is considered as an discrete version of a previous result for time-varying ordinary differential equations [1].

Keywords: Robust stability, Linear difference equation, Input-output operator, Stability radius

1. Introduction

Many control systems are subject to perturbations in terms of uncertain parameters. An important quantitative measure of stability robustness of a system to such perturbations is called the stability radius. The concept of stability radii was introduced by Hinrichsen and Pritchard 1986 for time-invariant differential (or difference) systems (see [2, 3]). It is defined as the smallest value ρ of the norm of real or complex perturbations destabilizing the system. If complex perturbations are allowed, ρ is called the complex stability radius. If only real perturbations are considered, the real radius is obtained. The computation of a stability radius is a subject which has attracted a lot of interest over recent decades, see e.g. [2, 3, 4, 5]. For further considerations in abstract spaces, see [6] and the references therein. Earlier results for time-varying systems can be found, e.g., in [1, 7]. The most successful attempt for finding a formula of the stability radius was an elegant result given by Jacob [1]. In that paper, it has been given by virtue of output-input operator a formula for L_p- stability for time-varying system subjected to additive structured perturbations of the form

$$\dot{x}(t) = B(t)x(t) + E(t)\Delta(F(\cdot)x(\cdot))(t),\ t \geqslant 0, x(0) = x_0,$$

where $E(t)$ and $F(t)$ are given scaling matrices defining the structure of the perturbation and Δ is an unknown disturbance. We now want to study a discrete version of this work by considering a difference equation with coefficients varying in time

$$x(n+1) = (A_n + E_n\Delta F_n)x(n),\ n \in \mathbb{N}. \tag{1}$$

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This problem has been studied by F. Wirth [8]. However, in this work, he has just given an estimate for stability radius. Following the idea in [1], we set up a formula for stability radius in the space l_p and show that when $p = 2$ and A, E, F are constant matrix, we obtain the result dealt with in [5]

The technique we use in this paper is somewhat similar to one in [1]. However, in applying the main idea of Jacob in [1] to the difference equations, we need some improvements. Many steps of the proofs in the paper [1] are considerably reduced and this reduction is valid not only in discrete case but also in continuous time one.

An outline of the remainder of the paper is as follows: the next section introduces the concept of Stability radius for difference equation in l_p. In Section 3 we prove a formula for computing the l_p- stability radius.

2. Stability radius for difference equation

We now establish a formulation for stability radius of the varying in times system

$$\begin{cases} x(n+1) = B_n x(n), & n \in \mathbb{N}, n > m \\ x(m) = x_0) \in \mathbb{R}^d. \end{cases} \tag{2}$$

It is easy to see that the equation (2) has a unique solution $x(n) = \Phi(n,m)x_0$ where $\Phi = \{\Phi(n,m)\}_{n \geqslant m \geqslant 0}$ is the Cauchy operator given by $\Phi(n,m) = B_{n-1} \cdots B_m, n > m$ and $\Phi(m,m) = I$. Suppose that the trivial solution of (2) is exponently stable, i.e., there exist positive constants M and $\alpha \in (0,1)$ such that

$$\|\Phi(n,m)\|_{\mathbb{K}^{d\times d}} \leqslant M\alpha^{n-m}, \quad n \geqslant m \geqslant 0. \tag{3}$$

We introduce some notations which are usually used later. Let X, Y be two Banach spaces and $\mathbb{N}$ be the set of all nonegative integer numbers. Put

- $l(0,\infty;X) = \{u : \mathbb{N} \to X\}$.
- $l_p(0,\infty;X) = \{u \in l(0,\infty;X) : \sum_{n=0}^{\infty} \|u(n)\|^p < \infty\}$ endowed with the norm $\|u\|_{l_p(0,\infty;X)} = (\sum_{n=0}^{\infty} \|u(n)\|^p)^{1/p} < \infty$.
- $l_p(s,t;X) = \{u \in l_p(0,\infty;X) : \ u(n) = 0 \text{ if } n \notin [s,t]\}$.
- $\mathrm{L}(l_p(0,\infty;X), l_p(0,\infty;Y))$ is the Banach space of all linear continuous operators from $l_p(0,\infty;X)$ to $l_p(0,\infty;Y)$.

Sometime, for the convenience of the formulation, we identify $l_p(s,t;X)$ with the space of all sequences $(u(n))_{n=s}^{t}$.

The truncated operators of $l(0,\infty;X)$ are defined by

$$\pi_t(x(\cdot))(k) = \begin{cases} x(k), & 0 \leqslant k \leqslant t, \\ 0, & k > t, \end{cases}$$

and

$$[x(\cdot)]_s(k) = \begin{cases} 0, & 0 \leqslant k < s, \\ x(k), & k \geqslant s. \end{cases}$$

An operator $\Gamma \in \mathrm{L}(l_p(0,\infty;X), l_p(0,\infty;Y))$ is said to be causal if $\pi_t A \pi_t = \pi_t A$ for any $t > 0$ (see [1]).

Let $A \in \L(l_p(0,\infty;\mathbb{K}^q), l_p(0,\infty;\mathbb{K}^s))$ be a causal operator. We consider the system (2) subjected to perturbation of the form

$$x(n+1) = B_n x(n) + E_n A(F_. x(\cdot))(n), \ \ n \in \mathbb{N}, \tag{4}$$

where $E_n \in \mathbb{K}^{d\times s}; F_n \in \mathbb{K}^{q\times d}$; the operator A is a perturbation.

A sequence $(y(n)) \in l(0,\infty;\mathbb{K}^d)$ is called a solution of (4) with the initial value $y(n_0) = x_0$ if

$$y(n+1) = B_n y(n) + E_n A([F_. y(\cdot)]_{n_0})(n), \ \ n \geqslant n_0. \tag{5}$$

Suppose that $(y(n))$ is a solution of (4) with the initial value $y(n_0) = x_0$. It is obvious that for $n > m > n_0$ the following constant-variation formula holds

$$y(n) = \Phi(n,m)y(m) + \sum_{m}^{n-1} \Phi(n,k+1)E_k A([\pi_{m-1}(F_. y(\cdot))]_{n_0})(k) + E_n A(\pi_{m-1}[F_. y(\cdot)]_{n_0})(n)$$
$$+ \sum_{m}^{n-1} \Phi(n,k+1)E_k A([F_. y(\cdot)]_m)(k) + E_n A([F_. y(\cdot)]_m)(n). \tag{6}$$

We are now in position to give a formula for stability radii for difference equation. Now let the unique solution to the initial value problem for (4) with initial value condition $x(n_0) = x_0$ denote by $x(\cdot\,; n_0, x_0)$. In the following, we suppose that

Hypothese 2.1. $E_n;\ F_n;$ are bounded on $\mathbb{N}$.

We define the following operators

$$(\mathbb{L}_0 u)(n) = F_n \textstyle\sum_{k=0}^{n-1} \Phi(n,k+1)E_k u(k)),$$
$$(\widehat{\mathbb{L}}_0 u)(n) = \textstyle\sum_{k=0}^{n-1} \Phi(n,k+1)E_k u(k),$$

for all $u \in l_p(0,\infty;\mathbb{K}^s),\ n > 0$. The first operator is called the input-output operator associated with (2). Put

$$(\mathbb{L}_{n_0} u)(n) = (\mathbb{L}_0[u]_{n_0})(n),\ (\widehat{\mathbb{L}}_{n_0} u)(n) = (\widehat{\mathbb{L}}_0[u]_{n_0})(n). \tag{7}$$

We see that these operators are independent of the choice of T_n. It is easy to verify the following auxiliary results.

Lemma 2.2. *Let* (3) *and Hypothesis hold. The following properties are true*

a) $\mathbb{L}_{n_0}, \in \mathcal{L}(l_p(n_0,\infty;\mathbb{K}^s), l_p(n_0,\infty;\mathbb{K}^q));\ \widehat{\mathbb{L}}_{n_0} \in \mathcal{L}(l_p(n_0,\infty;\mathbb{K}^s), l_p(n_0,\infty;\mathbb{K}^d))$,

b) $\|\mathbb{L}_t\| \leqslant \|\mathbb{L}_{t'}\|, \ \ t \geqslant t' \geqslant 0$,

c) There exist constants $M_1 \geqslant 0$ *such that*

$$\|\Phi(\cdot,n_0)x_0\|_{l_p(n_0,\infty;\mathbb{K}^d)} \leqslant M_1 \|x_0\|_{\mathbb{K}^d}, \ \ n_0 \geqslant 0,\ x_0 \in \mathbb{K}^d.$$

With these operators, any solution $x(n)$ having the initial condition $x(n_0) = x_0$) of (4) can be rewritten under the form

$$x(n) = \Phi(n,n_0)x_0 + \widehat{\mathbb{L}}_{n_0} A([F_. x(\cdot)]_{n_0})(n),\ n > n_0. \tag{8}$$

Definition 2.3. The trivial solution of (4) is said to be globally l_p-stable if there exist a constant $M_2 > 0$ such that

$$\|x(\cdot; n_0, x_0)\|_{l_p(n_0,\infty;\mathbb{K}^d)} \leqslant M_2 \|x_0\|_{\mathbb{K}^d}, \tag{9}$$

for all $x_0 \in \mathbb{K}^d$.

Remark 2.4. *From the inequality*

$$\|x(n;n_0,x_0)\|_{\mathbb{K}^d} \leqslant \|x(\cdot;n_0,x_0)\|_{l_p(n_0,\infty;\mathbb{K}^d)}$$

for any $n \geqslant n_0$*, it follows that that the global* l_p*-stability property implies the* $\mathbb{K}^d-$ *stability in initial condition.*

In comparing with [1, Definition 3.4]*, in the discrete case, we use only the relation* (9) *to define* l_p-*stability.*

3. A formula of the stability radius

First, the notion of the stability radius introduced in [1, 2, 9] is extended to time-varying difference system (2).

Definition 3.1. The complex (real) structured stability radius of (2) subjected to linear, dynamic and causal perturbation in (4) is defined by

$$r_{\mathbb{K}}(A;B,E,F) = \inf\{\|A\| : \text{ the trivial solution of (4) is not globally } l_p - \text{stable }\},$$

where $\mathbb{K} = \mathbb{C}, \mathbb{R}$, respectively.

Proposition 3.2. *If* $A \in \mathcal{L}(l_p(0,\infty;\mathbb{K}^q), l_p(0,\infty;\mathbb{K}^s))$ *is causal and satisfies*

$$\|A\| < \sup_{n_0 \geqslant 0} \|\mathbb{L}_{n_0}\|^{-1},$$

then the trivial solution of the system (4) *is globally* l_p- *stable.*

Proof. Let $m \geqslant n_0$ be arbitrarily given. It is easy to see that there exists an $M_3 > 0$ such that

$$\|x(n;n_0,x_0)\|_{\mathbb{K}^d} \leqslant M_3\,\|x_0\| \quad \forall\, n_0 \leqslant n \leqslant m. \tag{10}$$

Therefore,

$$\|x(\cdot,n_0,x_0)\|_{l_p(n_0,m,\mathbb{K}^d)} \leqslant (m-n_0)M_3\,\|x_0\|\,. \tag{11}$$

Now fix a number $m > n_0$ such that $\|A\|\,\|\mathbb{L}_m\| < 1$. Due to the assumption on $\|A\|$, such an m exists. It follows from (6) that

$$x(n,n_0,x_0) = \Phi(n,m)x(m,n_0,x_0) + \sum_{k=m}^{n-1}\Phi(n,k+1)E_k A([\pi_{m-1}(F_\cdot x(\cdot,n_0,x_0))]_{n_0})(k) + \sum_{k=m}^{n-1} E_k A([F_\cdot x(\cdot,n_0,x_0)]_m)(k)$$

for $n \geqslant m$. Therefore,

$$F_n x(n;n_0,x_0) = F_n\Phi(n,m)x(m;n_0,x_0) + (\mathbb{L}_m(A(\pi_{m-1}[Fx]_{n_0})))(n) + (\mathbb{L}_m(A([Fx]_m)))(n). \tag{12}$$

From (10) and (12) we have

$$\begin{aligned}
&\|F_\cdot x(\cdot;n_0,x_0)\|_{l_p(m,\infty,\mathbb{K}^q)} \leqslant \|F_\cdot\Phi(\cdot,m)x(m;n_0,x_0)\|_{l_p(m,\infty,K^q)} \\
&+ \|(\mathbb{L}_m(A(\pi_{m-1}[Fx]_{n_0})))(\cdot)\|_{l_p(m,\infty,\mathbb{K}^q)} + \|(\mathbb{L}_m(A([Fx]_m)))(\cdot)\|_{l_p(m,\infty,\mathbb{K}^q)} \\
&\leqslant M_1\|F_\cdot\|\,\|x(m;n_0,x_0)\|_{\mathbb{K}^d} \\
&+ \|\mathbb{L}_m\|\,\|A\|\,\|(\pi_{m-1}[Fx]_{n_0})(\cdot)\|_{l_p(n_0,m,\mathbb{K}^q)} + \|\mathbb{L}_m\|\,\|A\|\,\|[Fx]_m)(\cdot)\|_{l_p(m,\infty,\mathbb{K}^q)}\,.
\end{aligned}$$

Therefore,

$$(1-\|\mathbb{L}_m\|\,\|A\|)\,\|F_{\cdot}x(\cdot;n_0,x_0)\|_{l_p(m,\infty,\mathbb{K}^q)} \leqslant \|F_{\cdot}\|\big(M_1M_3+M_4\|\mathbb{L}_m\|\|A\|\big)\,\|x_0\|$$

which implies that

$$\|F_{\cdot}x(\cdot;n_0,x_0)\|_{l_p(m,\infty,\mathbb{K}^q)} \leqslant (1-\|\mathbb{L}_m\|\,\|A\|)^{-1}\,\|F_{\cdot}\|\big(M_1M_3+M_4\|\mathbb{L}_m\|\|A\|\big)\,\|x_0\|\,. \tag{13}$$

Setting $M_5 := (1-\|\mathbb{L}_m\|\,\|A\|)^{-1}\,\|F_{\cdot}\|\big(M_1M_3+M_4\|\mathbb{L}_m\|\|A\|\big)$ we obtain

$$\|F_{\cdot}x(\cdot;n_0,x_0)\|_{l_p(m,\infty,\mathbb{K}^q)} \leqslant M_5\,\|x_0\|_{\mathbb{K}^d}\,.$$

Hence, using (11) we have

$$\|F_{\cdot}x(\cdot;n_0,x_0)\|_{l_p(n_0,\infty,\mathbb{K}^q)} \leqslant M_6\,\|x_0\|_{\mathbb{K}^d}\,,$$

where $M_6 = M_4 + M_5$. Further, by (8)

$$\begin{aligned}&\|x(\cdot;n_0,x_0)\|_{l_p(n_0,\infty,\mathbb{K}^d)} \leqslant \|\Phi(\cdot,n_0)P_{n_0-1}x_0\|_{l_p(n_0,\infty,\mathbb{K}^d)} + \|\widehat{\mathbb{L}}_{n_0}\|\|A\|\|F_{\cdot}x(\cdot,n_0,x_0))\|_{l_p(n_0,\infty,\mathbb{K}^q)}\\ &\leqslant M_1\|P_{n_0-1}x_0\| + \|\widehat{\mathbb{L}}_{n_0}\|\|A\|\|F_{\cdot}x(\cdot,n_0,x_0))\|_{l_p(n_0,\infty,\mathbb{K}^q)} \leqslant M_7\,\|P_{n_0-1}x_0\|\,,\end{aligned}$$

where $M_7 = M_1 + \|\widehat{\mathbb{L}}_{n_0}\|\|A\|M_6$. The proof is complete.

Thus, by Proposition 4.3, the inequality

$$r_{\mathbb{K}}(A;B,E,F) \geqslant \sup_{n_0\geqslant 0}\|\mathbb{L}_{n_0}\|^{-1}$$

holds. We prove the converse relation.

We note that $\|L_n\|$ is decreasing in n. Therefore, there exists the limit

$$\lim_{n_0\to\infty}\|\mathbb{L}_{n_0}\|_{l_p(0,\infty;\mathbb{K}^q)} =: \frac{1}{\beta}.$$

Proposition 3.3. *For every* δ, $\beta < \delta < \|\mathbb{L}_0\|^{-1}$ *there exists a causal operator* $A \in \mathcal{L}(l_p(0,\infty;\mathbb{K}^q), l_p(0,\infty;\mathbb{K}^s))$ *with* $\|A\| < \delta$ *such that the trivial solution of* (4) *is not globally* l_p- *stable.*

Proof. Let us fix the numbers $\varepsilon > 0, \gamma > \beta$ satisfying $0 < \gamma(1-\varepsilon\gamma)^{-1} < A$. Since $\|\mathbb{L}_n\|_{l_p(0,\infty;\mathbb{K}^q)} \downarrow \frac{1}{\beta} > \frac{1}{\gamma}$,

$$\|\mathbb{L}_n\|_{l_p(0,\infty;\mathbb{K}^q)} > \frac{1}{\gamma}, \quad \forall n \geqslant 0.$$

In particular, $\|\mathbb{L}_0\| > \frac{1}{\gamma}$. Therefore, we can choose a function $\widetilde{f}_0 \in l_p(0,\infty;\mathbb{K}^s)$ with $\|\widetilde{f}_0\|_{l_p(0,\infty;\mathbb{K}^s)} = 1$ such that

$$\|\mathbb{L}_0\widetilde{f}_0\|_{l_p(0,\infty;\mathbb{K}^q)} > \frac{1}{\gamma}.$$

From the properties

$$\lim_{n\to\infty}\|\pi_n\widetilde{f}_0\|_{l_p(0,\infty;\mathbb{K}^s)} = 1, \quad \lim_{n\to\infty}\|\mathbb{L}_0\pi_n\widetilde{f}_0\|_{l_p(0,\infty;\mathbb{K}^q)} = \|\mathbb{L}_0\widetilde{f}_0\| > \frac{1}{\gamma},$$

it follows that there exists an $m_0 \in \mathbb{N}$ satisfying

$$\frac{1}{\|\pi_{m_0}\widetilde{f}_0\|}\|\mathbb{L}_0(\pi_{m_0}\widetilde{f}_0)\|_{l_p(0,\infty;\mathbb{K}^q)} > \frac{1}{\gamma}.$$

Denoting $f_0 = \frac{1}{\|\pi_{m_0}\widetilde{f}_0\|}\pi_{m_0}\widetilde{f}_0$ we obtain

$$\|f_0\|_{l_p(0,\infty;\mathbb{K}^s)} = 1, \quad \text{support } f_0 \subseteq [0,m_0] \quad \text{and} \quad \|\mathbb{L}_0f_0\|_{l_p(0,\infty;\mathbb{K}^q)} > \frac{1}{\gamma}.$$

Further, for any $n > m_0$ we have

$$\begin{aligned}\mathbb{L}_0(\pi_{m_0}h)(n) &= F_n \textstyle\sum_{k=0}^{m_0} \Phi(n,k+1)E_k(\pi_{m_0}h)(k)\\ &= F_n\Phi(n,m_0+1)\textstyle\sum_{k=0}^{m_0} \Phi(m_0+1,k+1)E_k(\pi_{m_0}h)(k).\end{aligned}$$

Therefore, by virtue of (3), there exists $n_0 > m_0$ such that

$$\|\mathbb{L}_0(\pi_{m_0}h)\|_{l_p(n_0,\infty;\mathbb{K}^q)} \leqslant \frac{\varepsilon}{2}\|h\|_{l_p(0,\infty;\mathbb{K}^s)}. \tag{14}$$

Similarly, we can find $n_0 < m_1 < n_1$ and f_1 satisfying

$$\|f_1\| = 1, \quad \text{support } f_1 \subseteq [n_0+1, m_1]$$

and

$$\|\mathbb{L}_0 f_1\|_{l_p(n_0+1,n_1;\mathbb{K}^q)} > \frac{1}{\gamma}, \quad \|\mathbb{L}_0(\pi_{m_1}h)\|_{l_p(n_1,\infty;\mathbb{K}^q)} \leqslant \frac{\varepsilon}{2^2}\|h\|_{l_p(0,\infty;\mathbb{K}^s)}.$$

Continuing this way, we can find the sequences (f_k) and $n_k \uparrow \infty$, $n_{k-1} < m_k < n_k$ having the following properties

$$\|f_k\|_{l_p(0,\infty;\mathbb{K}^s)} = 1, \quad \text{support } f_k \subseteq [n_{k-1}+1, m_k],$$

(with $n_{-1} = -1, m_{-1} = -1$) and

$$\|\mathbb{L}_0 f_k\|_{l_p(n_{k-1}+1,n_k;\mathbb{K}^q)} > \frac{1}{\gamma}, \quad \|\mathbb{L}_0(\pi_{m_k}h)\|_{l_p(n_k,\infty;\mathbb{K}^q)} \leqslant \frac{\varepsilon}{2^k}\|h\|_{l_p(0,\infty;\mathbb{K}^s)}. \tag{15}$$

Denote

$$\mathbb{Q}h = \sum_{k=0}^{\infty} 1_{[n_{k-1}+1,n_k]}\mathbb{L}_0([h]_{m_{k-1}+1}),$$

where 1_C denotes the indicator function of the set C. Let $f = \sum_{k=0}^{\infty} f_k$. By (15) we see that $\mathbb{L}_0 f \notin l_p(0,\infty;\mathbb{K}^q)$. Further,

- support $\mathbb{Q}f_k \subset [n_{k-1}+1, n_k]$,
- $\|(\mathbb{L}_0-\mathbb{Q})h\|_{l_p(0,\infty;\mathbb{K}^q)} \leqslant \sum_{k=1}^{\infty}\|\mathbb{L}_0(\pi_{m_{k-1}}h)\|_{l_p(n_{k-1},\infty;\mathbb{K}^q)} \leqslant \sum_{k=1}^{\infty}\frac{\varepsilon}{2^k}\|h\|_{l_p(0,\infty;\mathbb{K}^s)} = \varepsilon\|h\|_{l_p(0,\infty;\mathbb{K}^s)}$,

i.e.,

$$\|\mathbb{L}_0 - \mathbb{Q}\|_{l_p(0,\infty;\mathbb{K}^q)} \leqslant \varepsilon. \tag{16}$$

By Hahn-Banach theorem, for any $k \in \mathbb{N}$, there exists a linear functional, namely x_k^*, defined on $l_p(n_{k-1}+1, n_k, \mathbb{K}^q)$ such that

$$\|x_k^*\| = 1 \quad \text{and} \quad x_k^*\big(\mathbb{L}_0 f_k\big|_{n_{k-1}+1}^{n_k}\big) = \|\mathbb{L}_0 f_k\|_{l_p(n_{k-1}+1,n_k;\mathbb{K}^q)}.$$

We define a sequence of causal operators $A_k \in \mathcal{L}(l_p(0,\infty;\mathbb{K}^q), l_p(0,\infty;\mathbb{K}^s))$ by

$$A_k h = \frac{f_{k+1}}{\|\mathbb{L}_0 f_k\|_{l_p(n_{k-1}+1,n_k;\mathbb{K}^q)}} \cdot x_k^*(h\big|_{n_{k-1}+1}^{n_k}).$$

The sequence (A_k) has the following properties

- $A_k(\mathbb{L}_0 f_k) = A_k(\mathbb{Q}f_k) = f_{k+1}$,
- $\|A_k\| \leqslant \gamma$.

Let

$$\bar{A}h = \sum_{k=0}^{\infty} A_k h.$$

It is obvious

$$\|\bar{A}\| = \sup\{\|A_k\| : k \in \mathbb{N}\}.$$

Therefore, the operator $(I - (\mathbb{Q} - \mathbb{L}_0)\bar{A})$ is invertible and $\|(I - (\mathbb{Q} - \mathbb{L}_0)\bar{A})^{-1}\| \leqslant (1 - \varepsilon\gamma)^{-1}$. Set

$$A = \bar{A}(I - (\mathbb{Q} - \mathbb{L}_0)\bar{A})^{-1},$$
$$z = (I - (\mathbb{Q} - \mathbb{L}_0)\bar{A})\mathbb{Q}f.$$

We see that

$$\|A\| = \|\bar{A}_k(I - (\mathbb{Q} - \mathbb{L}_0)\bar{A})^{-1}\| \leqslant \gamma(1 - \varepsilon\gamma)^{-1} \leqslant \delta,$$

and

$$(I - \mathbb{L}_0\Delta)z = \big(I - (\mathbb{Q} - \mathbb{L}_0)\bar{A}\big)\mathbb{Q}f - \mathbb{L}_0\bar{A}\mathbb{Q}f = \mathbb{Q}(f - \bar{A}\mathbb{Q}f\big)$$
$$= \mathbb{Q}\left(f - \sum_{k=0}^{\infty}\Delta_k \sum_{i=0}^{\infty} 1_{[n_{i-1}+1,n_i]}\mathbb{L}_0([f]_{m_{i-1}+1})\right) = \mathbb{Q}f_0 = 1_{[0,n_0]}\mathbb{L}_0(f_0) =: g.$$

Hence,

$$(I - \mathbb{L}_0\Delta)z = g, \tag{17}$$

which implies that

$$(I - \widehat{\mathbb{L}}_0\Delta F)y = \widehat{\mathbb{L}}Ag, \tag{18}$$

where $y = \widehat{\mathbb{L}}Az$. From (18) we have $F_n y(n) = z(n)$ for any $n \geqslant n_0$. Therefore, $y \notin l_p(0, \infty; \mathbb{K}^q)$ because $z \notin l_p(0, \infty; \mathbb{K}^q)$ and F is bounded. Moreover, the relation (18) says that $y(\cdot)$ is a solution of the system

$$y(n+1) = B_n y(n) + E_n(\Delta(F_{\cdot} y(\cdot)))(n) + E_n(Ag)(n), \tag{19}$$

with the initial condition $y(0) = 0$. Put

$$h(n) := E_n(Ag)(n).$$

It is easy to see that $h(n)$ has a compact support. Substituting into the first one we obtain

$$y(n+1) = B_n y(n) + E_n A(F_{\cdot})y(\cdot))(n) + h(n). \tag{20}$$

For any $m \geqslant 0$, the equation

$$x(n+1) = B_n x(n) + E_n(\Delta(F_{\cdot} x(\cdot)))(n), \tag{21}$$

has a uniquely solution, say $x(\cdot, m, x_0)$, with the initial condition $x(m; m, x_0) = x_0$. We show that the sequence $(y(n))$ defined by

$$y(n+1) = \sum_{k=0}^{n} x(n+1, k+1, h(k)), \quad y(0) = 0. \tag{22}$$

is a solution of (20) with $y(0)=0$. Indeed,

$$
\begin{aligned}
y(n+1) &= \sum_{k=0}^{n} x(n+1,k+1,h(k)) = \sum_{k=0}^{n-1} x(n+1,k+1,h(k)) + h(n)\\
&= \sum_{k=0}^{n-1} B_n x(n,k+1,h(k)) + \sum_{k=0}^{n-1} E_n A(F_{\cdot} x(\cdot,k+1,h(k)))(n) + h(n)\\
&= B_n y(n,k+1,h(k)) + E_n A(F_{\cdot} \sum_{k=0}^{n-1} x(\cdot,k+1,h(k)))(n) + h(n)\\
&= B_n y(n,k+1,h(k)) + E_n A(F_{\cdot} \sum_{k=0}^{\cdot-1} x(\cdot,k+1,h(k)))(n) + h(n).
\end{aligned}
$$

Therefore,

$$
y(n+1) = B_n P_{n-1} y(n,k+1,h(k)) + E_n A((F_{\cdot} y(\cdot))))(n) + h(n),
$$

i.e., we get (20).

If (21) is globally l_p- stable, it follows that

$$
\begin{aligned}
\|y(\cdot)\|_{l_p(0,\infty;\mathbb{K}^d)} &= \left\{ \sum_{n=0}^{\infty} \left\| \sum_{k=0}^{n} x(n,k+1,h(k)) \right\|^p \right\}^{1/p}\\
&\leqslant \left\{ \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \|x(n;k+1,,h(k))\| \right)^p \right\}^{1/p}\\
&\leqslant \sum_{k=0}^{\infty} \left(\sum_{n=k+1}^{\infty} \|x(n;k+1,h(k))\|^p \right)^{1/p} \quad \text{(using Minkowski's inequality)}\\
&\leqslant M_{10} \sum_{k=0}^{\infty} \|h(k)\| < +\infty.
\end{aligned}
$$

Hence, it follows that

$$
\|y(\cdot)\|_{l_p(0,\infty;\mathbb{K}^d)} < \infty.
$$

That contradicts to $y(\cdot) \notin l_p(0,\infty;\mathbb{K}^d)$. This means that (4) is not globally stable.

Summing up we obtain.

Theorem 3.4. *For l_p-stability, the complex stability radius and real stability radius are equal and it is given by*

$$
r_{\mathbb{C}}(E,A;B,C) = r_{\mathbb{R}}(E,A;B,C) = \sup_{n_0 \geqslant 0} \|\mathbb{L}_{n_0}\|^{-1}.
$$

Corollary 3.5. *Let B, E, F be constant matrices and $p=2$. Then, there holds*

$$
r_{\mathbb{C}} = r_{\mathbb{R}} = \left\{ \sup_{|t| \geqslant 1} \left\| F\,(tI-B)^{-1}\,E \right\| \right\}^{-1}.
$$

Proof. Since B, E, F are constant matrices, we have

$$\left(\mathbb{L}_0 u\right)(n) = F\sum_{k=0}^{n-1}\Phi\left(n, k+1\right)Eu_k = F\sum_{k=0}^{n-1}\left(\prod_{m=n}^{k+1}B\right)Eu_k F\sum_{k=0}^{n-1}B^{n-k-1}Eu_k.$$

Denote by $H(h)$ the Fourier transformation of the function h. We see that

$$\begin{aligned}
H\left(\mathbb{L}_0 u\right) &= \sum_{n=0}^{\infty}\left(F\sum_{k=0}^{n-1}B^{n-k-1}Eu_k\right)e^{-in\omega} = \sum_{n=0}^{\infty}\left(F\sum_{k=0}^{n-1}B^{n-k-1}Eu_k\right)e^{-in\omega}\\
&= \sum_{k=0}^{\infty}F\left(\sum_{n=k}^{\infty}B^{n-k}e^{-i(n-k)\omega}\right)Eu_k e^{-ik\omega} = \sum_{k=0}^{\infty}F\left(e^{i\omega}I - B\right)^{-1}Eu_k e^{-ik\omega}\\
&= F\left(e^{i\omega}I - B\right)^{-1}E\sum_{k=0}^{\infty}u_k e^{-ik\omega} = F\left(e^{i\omega}I - B\right)^{-1}EH\left(u\right)\\
&= \left(F\left(e^{i\omega}I - B\right)^{-1}E\right)H\left(u\right) = F\left(\left(e^{i\omega}I - B\right)^{-1}\right)EH\left(u\right).
\end{aligned}$$

Therefore,

$$H\left(\mathbb{L}_0 u\right) = F\left(e^{i\omega}I - B\right)^{-1}EH\left(u\right).$$

Using Parseval equality we have

$$\|H\left(h\right)\| = \|h\|$$

for any $h \in l_2(0, \infty; \mathbb{K}^q)$. Hence,

$$\|\mathbb{L}_0 u\| = \|H\left(\mathbb{L}_0 u\right)\| = \left\|F\left(e^{i\omega}I - B\right)^{-1}E.H\left(u\right)\right\|.$$

Thus,

$$\begin{aligned}
\|\mathbb{L}_0\| &= \sup_{\|u\|\leqslant 1}\left\|F\left(e^{i\omega}I - B\right)^{-1}E.H\left(u\right)\right\|\\
&= \sup_{\|H(u)\|\leqslant 1}\left\|F\left(e^{i\omega}I - B\right)^{-1}E.H\left(u\right)\right\| = \sup_{\omega}\left\|F\left(e^{i\omega}I - B\right)^{-1}E\right\|.
\end{aligned}$$

Or

$$\|\mathbb{L}_0\| = \sup_{|t|=1}\left\|F\left(tI - B\right)^{-1}E\right\|.$$

Since $\lim_{t\to\infty}F\left(tA - B\right)^{-1}E = 0$,

$$r_{\mathbb{C}} = r_{\mathbb{R}} = \left\{\sup_{|t|\geqslant 1}\left\|F\left(tA - B\right)^{-1}E\right\|\right\}^{-1}.$$

The proof is complete.

Example 3.6. Calculate the stability radius of the unstructured system

$$X_{n+1} = \begin{pmatrix}-2 & 1\\ 1 & -1\end{pmatrix}X_n \qquad \forall n \geqslant 0. \tag{23}$$

The matrix $\begin{pmatrix}-2 & 1\\ 1 & -1\end{pmatrix}$ has two eigenvalues $\lambda_1 = 1/3$ and $\lambda_2 = 2/3$ which line in the unit ball. Therefore, the system (23) is asymptotically stable. Further

$$\left\|(tI - B)^{-1}\right\| = \begin{pmatrix}\frac{9t-2}{9t^2-9t+2} & -\frac{2}{9t^2-9t+2}\\ -\frac{2}{9t^2-9t+2} & \frac{9t-7}{9t^2-9t+2}\end{pmatrix}$$

We know that $\|(tI-B)^{-1}\|$ is the largest eigenvalue of $(tI-B)^{-1}\,(tI-B)^{-1}$ which is

$$\frac{-162t+162t^2+61+5\sqrt{324t^2-324t+97}}{2(81t^4-162t^3+117t^2-36t+4)}.$$

Hence,

$$\sup_{|t|=1}\left\|(tI-B)^{-1}\right\| = \sup_{|t|=1}\frac{-162t+162t^2+61+5\sqrt{324t^2-324t+97}}{2(81t^4-162t^3+117t^2-36t+4)} = \frac{61}{8}+\frac{5}{8}\sqrt{97}.$$

Thus,

$$r_{\mathbb{C}} = r_{\mathbb{R}} = \left(\frac{61}{8}+\frac{5}{8}\sqrt{97}\right)^{-1}.$$

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