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Dynamical behavior of Lotka–Volterra competition systems: non-autonomous bistable case and the effect of telegraph noise☆

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Abstract

This article is concerned with the study of trajectory behavior of Lotka–Volterra competition bistable systems and systems with telegraph noises. We proved that for bistable systems, there exists a unique solution, bounded above and below by positive constants. The oscillatory situation of systems with telegraph noises is pointed out.

MSC: 34C12; 60H10; 92D25

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1. Introduction

We consider the Lotka–Volterra system

$$\dot{x} = x(a(t) - b(t)x - c(t)y), \quad \dot{y} = y(d(t) - e(t)x - f(t)y), \tag{1.1}$$

where a, b, c, d, e, f are continuous functions. We suppose that a, b, c, d, e, f are bounded above and below by positive constants. This is a model of two competing species whose quantities at time t are $x(t)$ and $y(t)$. The functions a and d are the respective intrinsic growth rates; b and f measure the respective intraspecific competition within species x and y and the functions c, e measure the

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interspecific competitions between two species. The details of the ecological significance of such system are discussed in [5,8,10,11].

It is known that for Eq. (1.1) the quadrant plane $R^2_+=\{(u,v): 0<u<\infty; 0<v<\infty\}$ is invariant, i.e., if $(x(t),y(t))$ is a solution of (1.1) with $x(t_0)>0$, $y(t_0)>0$ for some $t_0\in R$ then $x(t)>0$, $y(t)>0$ for any $t\in(-\infty,\infty)$. In [3] it has been shown that under the condition:

$$\limsup_{|t|\to\infty}\frac{a(t)}{b(t)}<\liminf_{|t|\to\infty}\frac{d(t)}{e(t)}, \tag{1.2}$$

$$\limsup_{|t|\to\infty}\frac{d(t)}{f(t)}<\liminf_{|t|\to\infty}\frac{a(t)}{c(t)}, \tag{1.3}$$

Eq. (1.1) has a unique solution defined on $(-\infty,\infty)$ which is bounded above and below by positive constants. Furthermore, this unique solution has been proved to be attractive.

Conditions (1.2) and (1.3) which have been considered in [3] (see also [1,2]) ensure that the vector field of (1.1) always gets into the interior from the boundary of R^2_+. Therefore, it is easy to understand this unique bounded solution is attractive. We are now interested in the case where the inequalities (1.2),(1.3) are reversed. In this case, the vector field of (1.1) is bistable (see the illustration on the figure below). We can prove that there exists a forward neutral invariant curve such that any solution starting at a point on this curve is bounded below and above by positive constants on $[0,\infty)$. Such a curve also exists in the backward case. These curves intersect and thus, there is a (unique) solution starting at the common point of these curves, bounded below and above by positive constants on $(-\infty,\infty)$ (see Proposition 2.15). The other solutions must have a component tending to 0 as $t\to\infty$.

We first consider the deterministic system. On the other hand, the stochastic approach, versus the deterministic view is prevailing in the biological modeling, since it is natural to consider that the effect of the environmental or demographic randomness cannot be neglected on population dynamics. In the following, we introduce the stochastic effect on the above deterministic system in the form of the switching between two parameter sets. As an example of the biological meaning, the distinctive seasonal change such as dry and wet seasons are observed in monsoon forest, and it characterizes the vegetation there. The characteristic of some phenomena can be modeled with periodic or almost periodic functions. Also in boreal and arctic regions, seasonality exerts a strong influence on the dynamics of mammals, and indeed the model including this effect of seasonality by deterministic switching of parameters and equations has been proposed [9].

Hence, we will study on the trajectory behavior of System (1.1) when the coefficients of (1.1) at a time satisfy

$$\frac{a(t)}{b(t)}<\frac{d(t)}{e(t)},\quad \frac{d(t)}{f(t)}<\frac{a(t)}{c(t)}. \tag{1.4}$$

but at other time they satisfy the inequality

$$\frac{a(t)}{b(t)}>\frac{d(t)}{e(t)},\quad \frac{d(t)}{f(t)}>\frac{a(t)}{c(t)}. \tag{1.5}$$

This question perhaps is rather complicated. In this paper, we study only a special case where the coefficients of (1.1) depend on a telegraph noise, i.e., on a Markov process taking only two values.

Whenever the Markov process changes its state, the dynamics of System (1.1) switches between situations (1.4) and (1.5). It is proved that under a mild hypothesis, the solutions of (1.1) oscillate between the interior equilibrium point of the good case (1.4) and the boundary point of the bad system that satisfies (1.5).

The paper is organized as follows: In the second section, we study the non-autonomous systems satisfying the bistable condition. It is proved that there is a unique solution that is bounded above and below by positive constants. Section 3 is concerned with systems disturbed by telegraph noises. This is a mixing case between a stable system and a bistable one. We will point out the oscillatory situation of the solutions. In Section 4, the biological implications are discussed.

2. Non-autonomous Lotka–Volterra competition system under the bistable hypothesis

We now consider the Lotka–Volterra Eq. (1.1) with the following hypotheses:

Hypotheses 2.1. (1) There exist two constants m, M such that

$$m \leqslant g \leqslant M \quad \text{for any} \quad g := a, b, c, d, e, f.$$

$$(2)\ \liminf_{|t|\to\infty} \frac{a(t)}{b(t)} > \limsup_{|t|\to\infty} \frac{d(t)}{e(t)}, \tag{2.1}$$

$$\liminf_{|t|\to\infty} \frac{d(t)}{f(t)} > \limsup_{|t|\to\infty} \frac{a(t)}{c(t)}. \tag{2.2}$$

$$(3)\ \liminf_{|t|\to\infty} \frac{c(t)}{f(t)} > \limsup_{|t|\to\infty} \frac{b(t)}{e(t)}. \tag{2.3}$$

By virtue of Conditions (2.1) and (2.2), we can choose two constants k_1 and k_2 satisfying

$$\liminf_{|t|\to\infty} \frac{a(t)}{b(t)} > k_1 > \limsup_{|t|\to\infty} \frac{d(t)}{e(t)},$$

$$\liminf_{|t|\to\infty} \frac{d(t)}{f(t)} > k_2 > \limsup_{|t|\to\infty} \frac{a(t)}{c(t)}.$$

Therefore, there exist positive numbers γ and $t_0 > 0$ such that

$$\frac{a(t)}{b(t)} > k_1 + \gamma > k_1 - \gamma > \frac{d(t)}{e(t)}; \quad \frac{d(t)}{f(t)} > k_2 + \gamma > k_2 - \gamma > \frac{a(t)}{c(t)}, \tag{2.4}$$

for any t such that $|t| > t_0$.

We remark that under Conditions (2.1) and (2.2), System (1.1) is bistable. We illustrate this hypothesis by the following system (see Fig. 1):

$$\dot{x} = x(5 - x - 2y),$$

$$\dot{y} = y(3 - x - 0.5y).$$

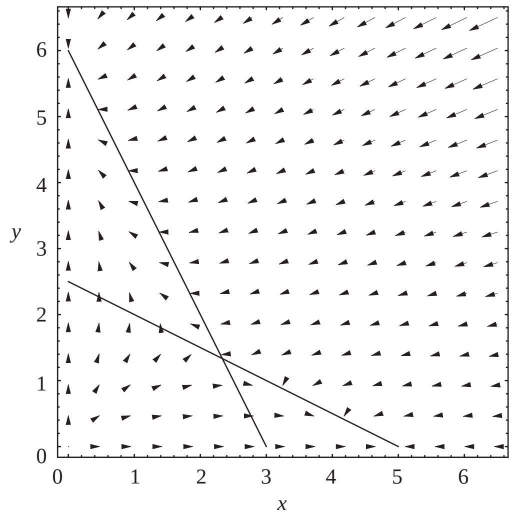

Fig. 1. The vector field of a bistable system.

First, we introduce some properties of the solutions of (1.1). The following lemma is known as preserving-order property of Lotka–Volterra system:

Proposition 2.2. *If* $(x_1(t), y_1(t))$ *and* $(x_2(t), y_2(t))$ *are two distinct solutions of* (1.1) *then for any* $t_0 \in R$ *we have the following*:

(a) *If* $x_1(t_0) \leqslant x_2(t_0)$; $y_1(t_0) \geqslant y_2(t_0)$ *then* $x_1(t) < x_2(t)$; $y_1(t) > y_2(t)$ *for all* $t > t_0$;
(b) *If* $x_1(t_0) \leqslant x_2(t_0)$; $y_1(t_0) \leqslant y_2(t_0)$ *then* $x_1(t) < x_2(t)$; $y_1(t) < y_2(t)$ *for all* $t < t_0$.

Proof. Item (a) is proved in [4, Lemma 4.4.1]. We prove the second assertion. Since the time reversed competition model is cooperative, which satisfies property (b). This is proved again in [4, Corollary 5.5.4]. □

Let $t_0 \in R$ arbitrary, we consider the forward equation of (1.1), i.e., for $t > t_0$.

Proposition 2.3. *Every solution* (x, y) *of Eq.* (1.1) *is bounded above on* $[t_0, +\infty)$.

Proof. From the inequality

$$\dot{x} = x(a - bx - cy) < x(a - bx) < x(M - mx),$$

it follows that

$$x(t) \leqslant \frac{x(t_0)\exp\{M(t - t_0)\}}{1 + x(t_0)m/M(\exp\{M(t - t_0) - 1)\}} < \max\left\{x(t_0), \frac{M}{m}\right\}. \tag{2.5}$$

Similarly,

$$y(t) \leqslant \max\left\{y(t_0), \frac{M}{m}\right\}. \qquad \square \tag{2.5'}$$

We will show that under Conditions (2.1) and (2.2), either every forward solution of Eq. (1.1) is strictly positive or it has a coordinate tending to 0 as $t \to \infty$. We first consider two "marginal" equations

$$\dot{u}(t) = u(t)[a(t) - b(t)u(t)], \tag{2.6}$$

$$\dot{v}(t) = v(t)[d(t) - f(t)v(t)]. \tag{2.6'}$$

Suppose that $u(t,s,x)$ is the solution of the Eq. (2.6) satisfying the initial conditions $u(s,s,x) = x$ and $v(t,s,y)$ is the solution of (2.6′) with $v(s,s,y) = y$.

Proposition 2.4. *For any $x \in R_+$, there exists a $T = T(x) > 0$ such that if $t > s > t_0$ and $t - s > T$ then*

$$u(t,s,x) > k_1 + \gamma/2; \quad v(t,s,x) > k_2 + \gamma/2. \tag{2.7}$$

Proof. While $u(t,s,x) \leqslant k_1 + \gamma/2$ we have from (2.4) $\dot{u} = u(a - bu) \geqslant u(a - b(k_1 + \gamma/2)) > u\gamma b/2 \geqslant u\gamma m/2$ that implies $u(t,s,x) \geqslant x \exp\{\gamma m(t - s)/2\}$. Similarly, while $v(t,s,x) \leqslant k_2 + \gamma/2$ we get $\dot{v} = v(d - fv) \geqslant v(d - f(k_2 + \gamma/2)) > v\gamma f/2 \geqslant v\gamma m/2$ which follows $v(t,s,x) \geqslant x \exp\{\gamma m(t - s)/2\}$. Therefore, we have only to choose $T = 2/\gamma m \max\{\ln(k_1 + \gamma/2)/x; \ln(k_2 + \gamma/2)/x\}$. □

We now turn to estimate solutions of (1.1). Denote by $(x(t,s,x_0,y_0), y(t,s,x_0,y_0))$ the solution of (1.1) satisfying $(x(s,s,x_0,y_0), y(s,s,x_0,y_0)) = (x_0,y_0)$.

Proposition 2.5. *There exist a neighborhood $\mathscr{U}$ of $R_+ \times \{0\}$ and a neighborhood $\mathscr{V}$ of $\{0\} \times R_+$ on $[0,\infty) \times [0,\infty)$ such that for any $s \geqslant t_0$*

(a) *if $(x_0,y_0) \in \mathscr{U}$ then*

$$\lim_{t\to\infty} y(t,s,x_0,y_0) = 0; \quad \lim_{t\to\infty} [x(t,s,x_0,y_0) - u(t,s,x_0)] = 0, \tag{2.8}$$

(b) *if $(x_0,y_0) \in \mathscr{V}$ then*

$$\lim_{t\to\infty} x(t,s,x_0,y_0) = 0; \quad \lim_{t\to\infty} [y(t,s,x_0,y_0) - v(t,s,y_0)] = 0. \tag{2.8'}$$

Proof. From inequalities (2.4) we can choose an $\alpha > 0$, $\epsilon > 0$ such that

$$a(t) - b(t)k_1 - c(t)\epsilon > \alpha > 0, \quad d(t) - e(t)k_1 < -\alpha,$$

$$d(t) - e(t)\epsilon - f(t)k_2 > \alpha > 0, \quad a(t) - c(t)k_2 < -\alpha, \tag{2.9}$$

for any $t : |t| > t_0$. Set

$$\mathscr{A}_1 = \{(u,v) : k_1 < u < \infty; 0 \leqslant v < \epsilon\},$$

$$\mathscr{A}_2 = \{(u,v) : k_2 < v < \infty; 0 \leqslant u < \epsilon\}.$$

From (2.9), we see that on $\{k_1\} \times [0,\epsilon]$ of the boundary of $\mathscr{A}_1$, $\dot{y} = y(d - ek_1 - fy) < y(d - ek_1) < -\alpha y$ and $\dot{x} = x(a - bx - cy) > x(a - bk_1 - c\epsilon) > \alpha x$. Further, on $(k_1,\infty) \times \{\epsilon\}$,

$\dot{y} = y(d - ex - fy) < y(d - ek_1) < -\alpha y$. Thus the vector field gets into $\mathscr{A}_1$. Hence, $\mathscr{A}_1$ is positively invariant. On the other hand, the inequality $\dot{y} < y(d - ek_1) < -\alpha y$ in $\mathscr{A}_1$ follows that $y(t, s, x_0, y_0) \downarrow 0$ as $t \to \infty$ for $(x_0, y_0) \in \mathscr{A}_1$. Therefore, by putting $z = 1/x - 1/u$ and from

$$\dot{z} = -az + \frac{cy}{x} \Rightarrow z(t) = \exp\left\{-\int_{t_0}^{t} a(s)\,\mathrm{d}s\right\}\left[z_0 + \int_{t_0}^{t} \exp\left\{\int_{t_0}^{s} a(u)\,\mathrm{d}u\right\}\frac{c(s)y(s)}{x(s)}\,\mathrm{d}s\right],$$

it follows that $\lim_{t\to\infty} z(t) = 0$ in noting that $x(t, s, x_0, y_0)$ is bounded below. Thus, by virtue of the upper bounded property of $x(t, s, x_0, y_0)$ and $u(t, s, x_0)$ on $[t_0, \infty)$ we get (2.8) for any $(x_0, y_0) \in \mathscr{A}_1$.

Let $x_0 > 0$, $T = T(x_0)$ be mentioned as in Proposition 2.4. We see that $(u(T(x_0) + s, s, x_0), 0) \in \mathscr{A}_1$ for any $s > t_0$. By the continuity of solution in initial conditions, there is a neighborhood U_{x_0} of $(x_0, 0)$ such that if $(x, y) \in U_{x_0}$ then $(x(T(x_0) + s, s, x, y), y(T(x_0) + s, s, x, y)) \in \mathscr{A}_1$. By virtue of the invariance property of $\mathscr{A}_1$ we follow (2.8) for any $(x, y) \in U_{x_0}$.

Put

$$\mathscr{U} = \bigcup_{x \in (0, k_1]} U_x \cup \mathscr{A}_1. \tag{2.10}$$

It is proved that $\mathscr{U}$ satisfies the requirement of (a). The construction of $\mathscr{V}$ for the item (b) is similar. The proof is completed. □

Corollary 2.6. *If* $\liminf_{t\to\infty} x(t) = 0$ *then* $\lim_{t\to\infty} x(t) = 0$. *Similarly, if* $\liminf_{t\to\infty} y(t) = 0$ *then* $\lim_{t\to\infty} y(t) = 0$.

Proof. Since $\liminf_{t\to\infty} x(t) = 0$, for a large t we have $(x(t), y(t)) \in \mathscr{V}$. The result then follows from Proposition 2.5 by paying attention $\mathscr{A}_1$ and $\mathscr{A}_2$ are positively invariant. □

We denote $\mathscr{A}$ the set of $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $\lim_{t\to\infty} y(t, t_0, x, y) = 0$ and $\mathscr{B}$ the set of (x, y) such that $\lim_{t\to\infty} x(t, t_0, x, y) = 0$.

Proposition 2.7. *$\mathscr{A}$ and $\mathscr{B}$ are open sets. Moreover for any $x_0 > 0$ and $y_0 > 0$, the sets $\mathscr{A} \cap \{\{x_0\} \times \mathbb{R}_+\}$ and $\mathscr{B} \cap \{\mathbb{R}_+ \times \{y_0\}\}$ are two open intervals.*

Proof. The fact that $\mathscr{A}$ and $\mathscr{B}$ are open follows from the continuity of the solution in the initial conditions. If $(x_0, y_0) \in \mathscr{A}$ then, by virtue of Proposition 2.2, with $0 < y < y_0$ we have $x(t, t_0, x_0, y_0) < x(t, t_0, x_0, y)$; $y(t, t_0, x_0, y) < y(t, t_0, x_0, y_0)$ for any $t > t_0$ which implies that $\lim_{t\to\infty} y(t, t_0, x_0, y) = 0$. Thus, $(x_0, y) \in \mathscr{A}$. The proof is similar for $\mathscr{B}$. □

Proposition 2.8. *On every line $x = x_0$, there exists at most one point (x_0, y_0) such that the solution starting at (x_0, y_0) at t_0 is bounded above and below by positive constants. A similar result can be formulated on every line $y = y_0$.*

Proof. By Proposition 2.2, we see that for the solution $(x(t), y(t))$, if $\lim_{t\to\infty} y(t) = 0$ then for every solution $(x_1(t), y_1(t))$ satisfying $x_1(t_0) = x(t_0)$; $y(t_0) > y_1(t_0)$ we have $\lim_{t\to\infty} y_1(t) = 0$. Similarly, if $\lim_{t\to\infty} x(t) = 0$ then for every solution $(x_1(t), y_1(t))$ satisfying $x_1(t_0) < x(t_0)$; $y(t_0) = y_1(t_0)$ we have $\lim_{t\to\infty} x_1(t) = 0$.

Suppose that $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are two solutions of (1.1) satisfying $\liminf_{t\to\infty} x_i(t) > 0$; $\liminf_{t\to\infty} y_i(t) > 0$ for $i=1,2$; $x_1(t_0)=x_2(t_0)=x_0$. Let $y_2(t_0) > y_1(t_0)$ then by Proposition 2.2, $x_2(t) < x_1(t)$ and $y_2(t) > y_1(t)$ for any $t > t_0$.

From (2.3) we follow that there exist three positive numbers α, β and γ such that

$$\frac{c(t)}{f(t)} > \frac{\alpha}{\beta} + \gamma, \quad \frac{b(t)}{e(t)} < \frac{\alpha}{\beta} - \gamma \quad \text{for all} \quad t \geqslant t_0. \tag{2.11}$$

Dividing both sides of (1.1) by x_2, x_1 and y_2, y_1, respectively, and subtracting yields

$$\frac{\dot{x}_2}{x_2} - \frac{\dot{x}_1}{x_1} = -b(x_2 - x_1) - c(y_2 - y_1),$$

$$\frac{\dot{y}_2}{y_2} - \frac{\dot{y}_1}{y_1} = -e(x_2 - x_1) - f(y_2 - y_1),$$

or, equivalently,

$$\left(\ln \frac{x_2}{x_1}\right)' = -b(x_2 - x_1) - c(y_2 - y_1),$$

$$\left(\ln \frac{y_2}{y_1}\right)' = -e(x_2 - x_1) - f(y_2 - y_1).$$

Putting

$$U(t) = \ln \frac{x_2(t)}{x_1(t)} \leqslant 0, \quad V(t) = \ln \frac{y_2(t)}{y_1(t)} \geqslant 0,$$

$$X(t) = x_2(t) - x_1(t) \leqslant 0, \quad Y(t) = y_2(t) - y_1(t) \geqslant 0,$$

we have

$$\dot{U}(t) = -b(t)X(t) - c(t)Y(t),$$

$$\dot{V}(t) = -e(t)X(t) - f(t)Y(t). \tag{2.12}$$

By multiplying the first equation of (2.12) by β and the second one by α and subtracting them we obtain

$$\beta\dot{U}(t) - \alpha\dot{V}(t) = (-\beta b(t) + \alpha e(t))X(t) + (-\beta c(t) + \alpha f(t))Y(t). \tag{2.13}$$

Since $U(t)$ and $V(t)$ are bounded above and below, it follows that

$$\int_{t_0}^{\infty} (-\beta b(s) + \alpha e(s))X(s)\,\mathrm{d}s + \int_{t_0}^{\infty} (-\beta c(s) + \alpha f(s))Y(s)\,\mathrm{d}s > -\infty.$$

From (2.11) it follows that $-\beta b(s)+\alpha e(s)$ and $\beta c(s)-\alpha f(s)$ are bounded below by positive constants. Therefore,

$$\int_{t_0}^{\infty} (-\beta b(s) + \alpha e(s))X(s)\,\mathrm{d}s > -\infty; \quad \int_{t_0}^{\infty} (\beta c(s) - \alpha f(s))Y(s)\,\mathrm{d}s < \infty.$$

Hence,

$$-\int_{t_0}^{\infty} X(s)\,\mathrm{d}s < \infty; \qquad \int_{t_0}^{\infty} Y(s)\,\mathrm{d}s < \infty.$$

Because $X(t)$ and $Y(t)$ and their derivatives are bounded then we get

$$\lim_{t\to\infty} X(t) = \lim_{t\to\infty} Y(t) = 0.$$

Furthermore, since $x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$ are also bounded below, we have

$$\lim_{t\to\infty} \frac{x_2(t)}{x_1(t)} = 1, \qquad \lim_{t\to\infty} \frac{y_2(t)}{y_1(t)} = 1$$

which implies that

$$\lim_{t\to\infty} U(t) = \lim_{t\to\infty} V(t) = 0. \tag{2.14}$$

Hence,

$$\begin{aligned} &\int_{t_0}^{\infty} b(t)X(t)\,\mathrm{d}t + \int_{t_0}^{\infty} c(t)Y(t)\,\mathrm{d}t = 0,\\ &\int_{t_0}^{\infty} e(t)X(t)\,\mathrm{d}t + \int_{t_0}^{\infty} f(t)Y(t)\,\mathrm{d}t \geqslant 0. \end{aligned} \tag{2.15}$$

If $\int_{t_0}^{\infty} b(s)X(s)\,\mathrm{d}s \neq 0$ and $\int_{t_0}^{\infty} c(s)Y(s)\,\mathrm{d}s \neq 0$ then by the mean value theorem of integrals it follows that

$$\frac{\int_{t_0}^{\infty} b(s)X(s)\,\mathrm{d}s}{\int_{t_0}^{\infty} e(s)X(s)\,\mathrm{d}s} \leqslant \sup_{t>t_0} \frac{b(t)}{e(t)} < \inf_{t>t_0} \frac{c(t)}{f(t)} \leqslant \frac{\int_{t_0}^{\infty} c(s)Y(s)\,\mathrm{d}s}{\int_{t_0}^{\infty} f(s)Y(s)\,\mathrm{d}s},$$

which contradicts to (2.15). Thus $\int_{t_0}^{\infty} b(s)X(s)\,\mathrm{d}s = 0$ and $\int_{t_0}^{\infty} c(s)Y(s)\,\mathrm{d}s = 0$. Hence, it is easy to see that $X(t) \equiv 0, Y(t) \equiv 0$. Proposition 2.8 is proved. □

Corollary 2.9. *If* $(x_1(t), y_1(t))$ *and* $(x_2(t), y_2(t))$ *are two solutions of* (1.1), *bounded above and below by positive constants then inequality* $x_1(t_0) < x_2(t_0)$ *implies the inequality* $y_1(t_0) < y_2(t_0)$.

Proof. Suppose to the contrary that $y_1(t_0) > y_2(t_0)$. We consider the solution $(x(t,x_3,y_3), y(t,x_3,y_3))$ with $x_3 = x_1(t_0)$; $y_3 = y_2(t_0)$. It follows from Proposition 2.2 that the solution $(x(t,x_3,y_3), y(t,x_3,y_3))$ is also bounded above and below by positive constants. This contradicts Proposition 2.8. □

Summing up, we have

Proposition 2.10. *There exists a number* $a > 0$ *and a strictly increasing, continuous function* $\varphi : [0,a) \to \mathbb{R}^+$ *such that every solution starting at* $(x, \varphi(x))$; $0 < x < a$ *at* t_0 *is bounded above and below by positive constants. Furthermore, for any* $(x_0, y_0) \notin$ *graph* φ, *either* $\lim_{t\to\infty} x(t,t_0,x_0,y_0) = 0$ *or* $\lim_{t\to\infty} y(t,t_0,x_0,y_0) = 0$.

Proof. Let a be the supremum of x_0 such that there exists a point (x_0, y_0) on the line $\{x_0\} \times \mathbb{R}_+$ satisfying $\lim_{t\to\infty} y(t, t_0, x_0, y_0) = 0$. For any $0 < x < a$ the set $\mathscr{A} \cap \{\{x\} \times \mathbb{R}_+\}$ is an interval, say $\{x\} \times (0, \bar{x})$. We put $\varphi(x) = \bar{x}$. It is easy to prove that φ is an increasing, continuous function defined on $[0, a)$ and $\lim_{x\to a} \varphi(x) = \infty$. □

We now proceed to study the behavior of solutions when $t \to -\infty$. Consider the backward system of (1.1)

$$\dot{x} = x(a(t) - b(t)x - c(t)y), \quad \dot{y} = y(d(t) - e(t)x - f(t)y), \quad t \leqslant -t_0.$$

Proposition 2.11. *If* $\limsup_{t\to-\infty} [x(t) + y(t)] = +\infty$ *then* $\lim_{t\to-\infty} x(t) = +\infty$ *and* $\lim_{t\to-\infty} y(t) = +\infty$

Proof. It follows from the fact that when either $x(t)$ or $y(t)$ is large we have

$$\dot{x} = x(a(t) - b(t)x - c(t)y) < \text{const} < 0, \quad \dot{y} = y(d(t) - e(t)x - f(t)y) < \text{const} < 0.$$

This implies that $x(t) \uparrow \infty$ and $y(t) \uparrow \infty$ as $t \downarrow -\infty$. □

Proposition 2.12. *There exist two negatively invariant open sets, namely* $\mathscr{U}_1$ *and* $\mathscr{V}_1$ *such that* $(0, k_1) \times \{0\} \subset \mathscr{U}_1$; $\{0\} \times (0, k_2) \subset \mathscr{V}_1$ *and if* $(x_0, y_0) \in \mathscr{U}_1$ *or* $(x_0, y_0) \in \mathscr{V}_1$ *then* $\lim_{t\to-\infty} x(t) = \lim_{t\to-\infty} y(t) = 0$.

Proof. Consider the equations

$$\dot{u} = u(-a + bu), \tag{2.16}$$

$$\dot{v} = v(-d + fv). \tag{2.17}$$

It is easy to see that if $u(t)$ is a solution of (2.16) satisfying $u(t_0) < k_1$ then $\lim_{t\to\infty} u(t) = 0$. So, by a similar way as in the proof of Proposition 2.5 we can construct an open set $\mathscr{U}_1$ by using the continuity of solutions on initial values. The construction of $\mathscr{V}_1$ is similar. □

Corollary 2.13. *If* $\liminf_{t\to-\infty} x(t) = 0$ *or* $\liminf_{t\to-\infty} y(t) = 0$ *then* $\lim_{t\to-\infty} x(t) = \lim_{t\to-\infty} y(t) = 0$.

Proof. Suppose $\liminf_{t\to-\infty} y(t) = 0$ then there exists a sequence $(\sigma_n) \downarrow -\infty$ such that

$$\lim_{n\to\infty} y(\sigma_n) = 0; \quad \dot{y}(\sigma_n) \leqslant 0.$$

Hence, $-d(\sigma_n) + e(\sigma_n)x(\sigma_n) + f(\sigma_n)y(\sigma_n) \leqslant 0$ which implies that $x(\sigma_n) < d(\sigma_n)/e(\sigma_n) < k_1$ for any n. Therefore, there is an $n \in N$ such that $(x(\sigma_n), y(\sigma_n)) \in \mathscr{U}_1$. By Proposition 2.12, we get $\lim_{t\to-\infty} x(t) = \lim_{t\to-\infty} y(t) = 0$. □

Summing up, we have:

Proposition 2.14. *There is a continuous strictly decreasing function* $\psi : [0, u_0] \to R_+$; $\psi(0) = v_0$ *and* $\psi(u_0) = 0$ *such that*

(1) *If* $\psi(x(t_0)) < y(t_0)$ *or* $x(t_0) > u_0$ *then* $\lim_{t\to-\infty} x(t) = \lim_{t\to-\infty} y(t) = \infty$.
(2) *If* $\psi(x(t_0)) > y(t_0)$ *and* $x(t_0) < u_0$ *then* $\lim_{t\to-\infty} x(t) = \lim_{t\to-\infty} y(t) = 0$.
(3) *If* $\psi(x(t_0)) = y(t_0)$ *then* $(x(t), y(t))$ *is bounded above and below by positive constants on* $(-\infty, -t_0]$.

Proof. It is easy to show that there exists $u_0 > 0$ such that if $u(t)$ is the solution of (2.16) with $u(t_0) < u_0$ then $\lim_{t\to\infty} u(t) = 0$ and with $u(t_0) > u_0$ then $\lim_{t\to\infty} u(t) = +\infty$. Similarly, there exists $v_0 > 0$ such that if $v(t)$ is the solution of (2.17) with $v(t_0) < v_0$ then $\lim_{t\to\infty} v(t) = 0$ or with $v(t_0) > v_0$ then $\lim_{t\to\infty} v(t) = +\infty$. On the other hand, by a similar argument as in the proof of Proposition 2.8, it follows that on every line $x = x_0$ with $0 < x_0 < u_0$, there is exactly one (x_0, y_0) such that the solution starting at (x_0, y_0) is bounded above and below by positive constants. We put $\psi(x_0) = y_0$. It is easy to check that ψ is the desired function. □

By combining Propositions 2.10 and 2.14 we obtain

Proposition 2.15. *Under conditions* (2.1)–(2.3), *System* (1.1) *has a unique solution bounded above and below by positive constants on* $(-\infty, \infty)$.

Proof. By Proposition 2.10 we see that for every $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$, either $(x(t_0, 0, x, y),\ y(t_0, 0, x, y)) \in \mathscr{A}$ or $(x(t_0, 0, x, y), y(t_0, 0, x, y)) \in \mathscr{B}$ or $(x(t_0, 0, x, y), y(t_0, 0, x, y)) \in graph(\varphi)$. Let us define

$$\mathscr{A}^* = \{(x, y) : (x(t_0, 0, x, y), y(t_0, 0, x, y)) \in \mathscr{A}\},$$

$$\mathscr{B}^* = \{(x, y) : (x(t_0, 0, x, y), y(t_0, 0, x, y)) \in \mathscr{B}\}.$$

By Proposition 2.2, the sets $\mathscr{A}^*$ and $\mathscr{B}^*$ have the similar properties mentioned in Proposition 2.7. Moreover, on every line $\{x\} \times \mathbb{R}_+$, there is at most a point (x, y) such that $(x(t_0, 0, x, y), y(t_0, 0, x, y)) \in graph(\varphi)$. Thus, there exist a number a^* and a function $\varphi^* : [0, a^*) \to \mathbb{R}_+$ which has the same property as φ. Similarly, there exists a function ψ^* having the similar properties as ψ. Since φ^* is increasing and ψ^* is decreasing then there is a unique point $(x^*, y^*) = graph(\varphi^*) \cap graph(\psi^*)$. The solution starting at (x^*, y^*) is bounded above and below on $(-\infty, +\infty)$. The proposition is proved. □

In case the coefficients a, b, c, d, e, f are constant, we can go further the results in Proposition 2.15. Conditions (2.1) and (2.2) now become $a/b > d/e$; $a/c < d/f$. Condition (2.3) is followed from (2.1) and (2.2). We will show that in this case the function φ^* in Proposition 2.15 is defined on $[0, \infty]$ and $\lim_{x\to\infty} \varphi^*(x) = \infty$.

We point out that on every line $x = x_0$ there exists at least a point (x_0, y_0) such that $\lim_{t\to\infty} x(t, x_0, y_0) = 0$. Let us take a point $(x_1, y_1) \in \mathscr{A}_2$ with $y_1 > d/f$ (see the definition of $\mathscr{A}_2$ given in the proof of Proposition (2.5)). By virtue of the Proposition 2.14, $\lim_{t\to-\infty} x(t, x_1, y_1) = \infty$;

$\lim_{t\to-\infty} y(t,x_1,y_1)=\infty$. Thus, there is $T_0>0$ such that $x(-T_0,x_1,y_1)=x_0$. Set $y_0=y(-T_0,x_1,y_1)$. Denote $\Phi(t,x,y)=(x(t,x,y),y(t,x,y))$. By dynamic property of Φ we obtain

$$\Phi(T_0,x_0,y_0)=\Phi(T_0,\Phi(-T_0,x_1,y_1))=\Phi(0,x_1,y_1)=(x_1,y_1),$$

i.e., the solution starting at (x_0,y_0) will be in $\mathscr{A}_2$ at the time T_0. Since $\mathscr{A}_2$ is positively invariant, and if $(x_1,y_1)\in\mathscr{A}_2$ then $\lim_{t\to\infty} x(t,x_1,y_1)=0$, we get $\lim_{t\to\infty} x(t,x_0,y_0)=0$. Similarly, we can prove on the line $y=y_0$, there is a point (x_0,y_0) such that $\lim_{t\to\infty} y(t,x_0,y_0)=0$. Thus φ^* is defined on $[0,\infty)$ and $\lim_{x\to\infty}\varphi^*=\infty$.

By a similar argument, we can point out that the function ψ^* in Proposition 2.15 is defined on $[0,a/b)$ and its range is $[0,d/f)$.

Remark. It is easy to see that, in fact, the graph of φ^* and of ψ^* are respectively the stable manifold and unstable manifold of system (1.1).

We deal with an estimate of the vanish time of the solution of (1.1). Denote by ℓ the graph of φ^* and

$$\mathscr{A}=\{(x,y): y<\varphi(x), x>0\},$$

$$\mathscr{B}=\{(x,y): y>\varphi(x), x>0\}.$$

The above results say that if $(x,y)\in\mathscr{A}$ then $\lim_{t\to\infty} y(t,x,y)=0$; $\lim_{t\to\infty}(x(t,x,y)-u(t,x))=0$ and if $(x,y)\in\mathscr{B}$ then $\lim_{t\to\infty} x(t,x,y)=0$; $\lim_{t\to\infty}(y(t,x,y)-v(t,x))=0$ where u and v are the solution of (2.6) and (2.6′).

Lemma 2.16. *For any compact set $K\subset\mathscr{A}$ (respect. $K\subset\mathscr{B}$) and any ϵ-neighborhood U_ϵ of $(a/b,0)$ (respect. V_ϵ neighborhood of $(0,d/f)$), there is a $T_1^*>0$ (respect. $T_2^*>0$) such that $\Phi(t,x,y)\in U_\epsilon$ for any $t>T_1^*$ (respect. $\Phi(t,x,y)\in V_\epsilon$ for any $t>T_2^*$) and any $(x,y)\in K$.*

Proof. We need to consider the case of small ϵ such that $U_\epsilon\subset\mathscr{U}$ (see (2.10)). Let $K\subset\mathscr{A}$ and $(x,y)\in K$, then by Proposition 2.10, there is a $T=T(x,y)$ such that $\Phi(t,x,y)\in U_\epsilon$ $\forall t\geqslant T(x,y)$. By the continuity of the solution in the initial conditions, there exists an open neighborhood $U_{x,y}$ of (x,y) such that $\Phi(t,u,v)\in U_\epsilon$ $\forall t\geqslant T(x,y)$ for any $(u,v)\in U_{x,y}$. The family $(U_{x,y})_{(x,y)\in K}$ is an open covering of K. Since K is compact, there are $U_{x_i,y_i}, i=1,\dots,n$ such that $K\subset\bigcup_{i=1}^n U_{x_i,y_i}$. Put $T_1^*=\max_{1\leqslant i\leqslant n} T(x_i,y_i)$ we have the result. The case $K\subset\mathscr{B}$ is proved by a similar way. □

3. Lotka–Volterra competition systems under the telegraph noises

Let us consider a Markov process $(\xi_t)_{t\geqslant 0}$, defined on the probability space $(\Omega,\mathscr{F},\mathbf{P})$, with values in the set of two elements, say $E=\{+,-\}$. Suppose that (ξ_t) has the transition intensities $+\xrightarrow{\alpha}-$ and $-\xrightarrow{\beta}+$ with $\alpha>0$, $\beta>0$. The process (ξ_t) has a unique stationary distribution

$$p=\lim_{t\to\infty}\mathbf{P}\{\xi_t=+\}=\frac{\beta}{\alpha+\beta};\quad q=\lim_{t\to\infty}\mathbf{P}\{\xi_t=-\}=\frac{\alpha}{\alpha+\beta}. \tag{3.1}$$

The trajectory of (ξ_t) is piecewise constant, cadlag function. Suppose that

$$0=\tau_0<\tau_1<\tau_2<\cdots<\tau_n<\cdots$$

are its jump times. Put

$$\sigma_1 = \tau_1 - \tau_0, \sigma_2 = \tau_2 - \tau_1, \ldots, \sigma_n = \tau_n - \tau_{n-1} \ldots .$$

It is known that, if ξ_0 is given, (σ_n) is a sequence of independent random variables. Moreover, if $\xi_0 = +$ then σ_{2n+1} has the exponential density $\alpha 1_{[0,\infty)} \exp(-\alpha t)$ and σ_{2n} has the density $\beta 1_{[0,\infty)} \exp(-\beta t)$. Conversely, if $\xi_0 = -$ then σ_{2n} has the exponential density $\alpha 1_{[0,\infty)} \exp(-\alpha t)$ and σ_{2n+1} has the density $\beta 1_{[0,\infty)} \exp(-\beta t)$ (see [7, Vol. 2, pp. 217]). Here $1_{[0,\infty)} = 1$ for $t \geqslant 0$ (=0 for $t < 0$). As a consequence of this property we have

Lemma 3.1. *For any $s_2 > s_1 > 0$, $t_2 > t_1 > 0$ the event*

$$\{\sigma_{2n} \in [s_1, s_2]; \sigma_{2n+1} \in [t_1, t_2]; \text{ i.o. } n > 0\}$$

has a probability one (i.o. means "infinitely often").

Proof. Since

$$D := \{\sigma_{2n} \in [s_1, s_2]; \sigma_{2n+1} \in [t_1, t_2]; \text{ i.o. } n > 0\}$$
$$= \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \{\sigma_{2n} \in [s_1, s_2]; \sigma_{2n+1} \in [t_1, t_2]\},$$

then,

$$\mathbf{P}\{D\} = \lim_{k \to \infty} \bigcup_{n=k}^{\infty} \{\sigma_{2n} \in [s_1, s_2]; \sigma_{2n+1} \in [t_1, t_2]\}.$$

On the other hand, given $\xi_0 = +$, it is easy to see that

$$\mathbf{P}\{\sigma_{2n} \in [s_1, s_2]; \sigma_{2n+1} \in [t_1, t_2]\} = (e^{-\beta s_1} - e^{-\beta s_2})(e^{-\alpha t_1} - e^{-\alpha t_2}) > 0.$$

Hence, by Kolmogorov's $0 - 1$ law, we follow the result. □

We now consider the competition equation

$$\dot{x} = x(a(\xi_t) - b(\xi_t)x - c(\xi_t)y), \quad \dot{y} = y(d(\xi_t) - e(\xi_t)x - f(\xi_t)y), \tag{3.2}$$

where $g : E \to \mathbb{R}_+$ for $g = a, b, c, d, e, f$.

We study two marginal equations

$$\dot{u} = u(a(\xi_t) - b(\xi_t)u), \tag{3.3}$$

$$\dot{v} = v(d(\xi_t) - f(\xi_t)v). \tag{3.4}$$

To simplify notations, we put

$$h^+ = h^+(u) = u(a(+) - b(+)u); \quad h^- = h^-(u) = u(a(-) - b(-)u),$$

$$g^+ = g^+(v) = v(d(+) - f(+)v); \quad g^- = g^-(v) = v(d(-) - f(-)v).$$

Suppose that $u(t)$ is a solution of (3.3) and $v(t)$ is a solution of (3.4). The processes $(\xi_t, u(t))$ and $(\xi_t, v(t))$ are Markov with the respective infinitesimal operators

$$L_1\ell(i,u) = \begin{cases} -\alpha(\ell(+,u)-\ell(-,u)) + h^+(u)\dfrac{d}{du}\ell(+,u) & \text{if } i=+, \\ \beta(\ell(+,u)-\ell(-,u)) + h^-(u)\dfrac{d}{du}\ell(-,u) & \text{if } i=-, \end{cases}$$

$$L_2\ell(i,v) = \begin{cases} -\alpha(\ell(+,v)-\ell(-,v)) + g^+(v)\dfrac{d}{dv}\ell(+,v) & \text{if } i=+, \\ \beta(\ell(+,v)-\ell(-,v)) + g^-(v)\dfrac{d}{dv}\ell(-,v) & \text{if } i=-, \end{cases}$$

with $\ell(i,x)$ to be a function defined on $E\times(0,\infty)$, continuously differentiable in x. The stationary density (μ^+,μ^-) of $(\xi_t,u(t))$ can be found from the Fokker–Planck equation

$$-\alpha\mu^+(u) + \beta\mu^-(u) - \frac{d}{du}[h^+\mu^+(u)] = 0, \tag{3.5}$$

$$\alpha\mu^+(u) - \beta\mu^-(u) - \frac{d}{du}[h^-\mu^-(u)] = 0, \tag{3.6}$$

with $\mu^+(u)\geqslant 0$; $\mu^-(u)\geqslant 0$ and

$$\int_{[a(+)/b(+),a(-)/b(-)]} (p\mu^+(u) + q\mu^-(u))\,du = 1.$$

On adding (3.5) and (3.6) we obtain

$$\frac{d}{du}[h^+\mu^+ + h^-\mu^-] = 0.$$

Thus,

$$h^+\mu^+ + h^-\mu^- = m = \text{const}.$$

Or

$$\mu^- = \frac{m - h^+\mu^+}{h^-}\cdot \tag{3.7}$$

Substituting (3.7) into (3.5) we have

$$-\alpha\mu^+(u) + \beta\frac{m-h^+\mu^+}{h^-} - \frac{d}{du}[h^+\mu^+(u)] = 0.$$

Denote $X = h^+(u)\mu^+(u)$ then we come to the equation

$$\frac{dX}{du} + \left(\frac{\alpha}{h^+(u)} + \frac{\beta}{h^-(u)}\right)X = \frac{\beta m}{h^-(u)}.$$

This equation has the general solution

$$X(u) = F(u)\left[X(u_0) + \beta m\int_{u_0}^{u}\frac{1}{F(v)h^-(v)}\,dv\right],$$

where

$$F(u)=\exp\left\{-\int_{u_0}^{u}\left(\frac{\alpha}{h^+(x)}+\frac{\beta}{h^-(x)}\right)\,\mathrm{d}x\right\},$$

and u_0 is chosen in $(a(+)/b(+),a(-)/b(-))$. Thus,

$$\mu^+(u)=\frac{F(u)}{h^+(u)}\left[X(u_0)+\beta m\int_{u_0}^{u}\frac{1}{F(x)h^-(x)}\,\mathrm{d}x\right],$$

$$\mu^-(u)=\frac{F(u)}{h^-(u)}\left[m-X(u_0)+\alpha m\int_{u_0}^{u}\frac{1}{F(x)h^+(x)}\,\mathrm{d}x\right]. \tag{3.8}$$

The constants m and $X(u_0)$ are chosen such that

$$\mu^+(u)\geqslant 0,\quad \mu^-(u)\geqslant 0,\quad \int_{[a(+)/b(+),a(-)/b(-)]}(p\mu^+(u)+q\mu^-(u))\,\mathrm{d}u=1.$$

Similarly, we can compute the stationary density (v^+,v^-) of the process $(\xi_t,v(t))$ which is given by

$$v^+(v)=\frac{G(v)}{g^+(v)}\left[Y_0+\beta n\int_{v_0}^{v}\frac{1}{G(t)g^-(t)}\,\mathrm{d}t\right],$$

$$v^-(v)=\frac{G(v)}{g^-(v)}\left[n-Y_0+\alpha n\int_{v_0}^{v}\frac{1}{G(t)g^+(t)}\,\mathrm{d}t\right], \tag{3.9}$$

where,

$$G(v)=\exp\left\{-\int_{v_0}^{v}\left(\frac{\alpha}{g^+(x)}+\frac{\beta}{g^-(x)}\right)\,\mathrm{d}x\right\},$$

and v_0 is chosen in $(d(+)/f(+),d(-)/f(-))$. The constants n and Y_0 are chosen such that

$$v^+(v)\geqslant 0,\quad v^-(v)\geqslant 0,\quad \int_{[d(+)/f(+),d(-)/f(-)]}(pv^+(v)+qv^-(v))\,\mathrm{d}v=1.$$

In fact we can calculate the explicit formula for the stationary densities $\mu^+(u),\mu^-(u)$, $v^+(v),v^-(v)$ but in practice, it is not useful. To study some of their properties, we use the simulation method instead.

Proposition 3.2. (a) *If*

$$\lambda:=\int_{[d(+)/f(+),d(-)/f(-)]}(p(a(+)-c(+)v)v^+(v)+q(a(-)-c(-)v)v^-(v))\,\mathrm{d}v>0, \tag{3.10}$$

then $\limsup_{t\to\infty}\frac{1}{t}\int_0^t x(s,x,y)\,\mathrm{d}s>0$ *for P–a.s. and for any* $x>0,\ y>0$.

(b) *If*

$$\theta:=\int_{[a(+)/b(+),a(-)/b(-)]}(p(d(+)-e(+)u)\mu^+(u)+q(d(-)-e(-)u)\mu^-(u))\,\mathrm{d}u>0, \tag{3.11}$$

then $\limsup_{t\to\infty}\frac{1}{t}\int_0^t y(s,x,y)\,\mathrm{d}s>0$ *for P-a.s. and for any* $x>0,\ y>0$.

Here $(x(t,x,y),y(t,x,y))$ *is a solution of* (3.2).

Proof. (a) We remark that if $y(0)=v(0)$ then the inequality $\dot{y}=y(d-ex-fy)\leqslant y(d-fy)$ implies that $y(t)\leqslant v(t)$ for any $t>0$ by comparison principle. Hence,

$$\frac{\dot{x}(t)}{x(t)}=a(\xi_t)-b(\xi_t)x(t)-c(\xi_t)y(t)\geqslant a(\xi_t)-b(\xi_t)x(t)-c(\xi_t)v(t).$$

Then,

$$\frac{1}{t}\int_0^t b(\xi_s)x(s)\,\mathrm{d}s+\frac{\ln x(t)-\ln x(0)}{t}\geqslant\frac{1}{t}\int_0^t(a(\xi_s)-c(\xi_s)v(s))\,\mathrm{d}s.$$

Since $(\xi_t,v(t))$ is a Markov process having a unique invariant measure $(v^+(v),v^-(v))$ then by the law of large numbers:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t(a(\xi_s)-c(\xi_s)v(s))\,\mathrm{d}s$$
$$=\int_{[d(+)/f(+),d(-)/f(-)]}(p(a(+)-c(+)v)v^+(v)+q(a(-)-c(-)v)v^-(v))\,\mathrm{d}v=\lambda>0.$$

(see [6, Lemma 3.1]). Moreover, $\limsup_{t\to\infty}\frac{(\ln x(t)-\ln x(0))}{t}\leqslant 0$, hence,

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t b(\xi_s)x(s)\,\mathrm{d}s\geqslant\lambda.$$

Since $b(\xi_t)$ is bounded above by M then it follows that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x(s)\,\mathrm{d}s\geqslant\frac{\lambda}{M}.$$

Similarly,

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t y(s)\,\mathrm{d}s\geqslant\frac{\theta}{M}.$$

The proof is completed. □

Remark. From the proof of Proposition 3.2 we see that

$$\limsup_{t\to\infty}x(t)\geqslant\frac{\lambda}{pb(+)+qb(-)}.$$

and

$$\limsup_{t\to\infty}y(t)\geqslant\frac{\theta}{pf(+)+qf(-)}.$$

To get the further information on the trajectory behavior of the solutions of (3.2), we need the following hypotheses:

Hypotheses 3.3. The coefficients of Eq. (3.2) satisfy:

(a) $a(+)/b(+)<d(+)/e(+);\quad a(+)/c(+)>d(+)/f(+),$ (3.12)

(b) $a(-)/b(-)>d(-)/e(-);\quad a(-)/c(-)<d(-)/f(-).$ (3.13)

Inequalities (3.12) are only a special case of (1.2), (1.3) and inequalities (3.13) are of (2.1) and (2.2). Thus, they ensure the existence of a unique rest point $(x^+, y^+) > 0$ such that every solution $(x^+(t,x,y), y^+(t,x,y))$ with $(x^+(0,x,y), y^+(0,x,y)) = (x,y) > 0$ of the system

$$\dot{x}^+(t) = x^+(t)(a(+) - b(+)x^+(t) - c(+)y^+(t)),$$
$$\dot{y}^+(t) = y^+(t)(d(+) - e(+)x^+(t) - f(+)y^+(t)), \tag{3.14}$$

satisfies $\lim_{t\to\infty}(x^+(t), y^+(t)) = (x^+, y^+)$. We estimate the time when the solution $(x^+(t), y^+(t))$ enters in a neighborhood of (x^+, y^+).

Since whenever $x(t)$ and $y(t)$ are small (respectively: whenever at least one of $x(t)$ or $y(t)$ is large), $x(t)\uparrow$ and $y(t)\uparrow$ (respect. $x(t)\downarrow$ and $y(t)\downarrow$) then there are two constants k_3, M satisfying $0 < k_3 < \min\{x^+, y^+, x^-, y^-\} < M$ ((x^-, y^-) is a unique solution of the equation $a(-) - b(-)x - c(-)y = 0$; $d(-) - e(-)x - f(-)y = 0$) and $t_0 > 0$ such that $x(t) < M, y(t) < M$ and either $x(t) \geqslant k_3$ or $y(t) \geqslant k_3$ for any $t \geqslant t_0$. Here $(x(t), y(t))$ is a solution of (3.2). Therefore, without loss of generality, we suppose that $x(t) < M, y(t) < M$ and either $x(t) \geqslant k_3$ or $y(t) \geqslant k_3$ for any $t \geqslant 0$.

Lemma 3.4. *For any small $\delta > 0$, $\delta_1 > 0$, there exists a $T_3^* = T_3^*(\delta, \delta_1) > 0$ such that $(x^+(t), y^+(t)) \in U_{\delta_1}$ for any $t \geqslant T_3^*$, provided that $\delta < x^+(0) < M$, $\delta < y^+(0) < M$. Here U_{δ_1} is the δ_1-neighborhood of (x^+, y^+).*

Proof. The proof can be done by a similar way as in the proof of Lemma 2.16 and we omit it here. □

Lemma 3.5. *There exists ϵ_1 ($k_3 > \epsilon_1 > 0$) such that for any $0 < t_1 < t_2$, if $(x^+(t_1), y^+(t_1)) \in [k_3, \infty) \times [0, \epsilon_1]$ then $x^+(t) \geqslant k_3\ \forall t > t_1$. Moreover, if $y^+(t_2) < \epsilon_1$ then $\sup_{t_1 < t < t_2} y^+(t) < \epsilon_1$. There is a similar result for the case $(x^+(t_1), y^+(t_1)) \in [0, \epsilon_1] \times [k_3, \infty)$.*

Proof. When $y^+(t_1)$ is small, if $x^+(t_1) > a(+)/b(+)$ then either $y^+(t)\uparrow$ for $t > t_1$ or $y^+(t)$ has a unique extremal point on (t_1, t_2) that is the minimum one. Further, if $x^+(t_1) \leqslant a(+)/b(+)$, by the continuity of the solution in the initial conditions we can find $\epsilon_1 > 0$ such that if $x^+(t_1) \geqslant k_3, 0 < y^+(t_1) < \epsilon_1$ then the orbit of the solution $(x^+(t), y^+(t))$ will hit the interval $[(x^+, y^+), (a(+)/b(+), 0)]$ at a $t \geqslant t_1$. Thus $y^+(t)\uparrow$ for $t > t_1$. In any case we always obtain $\sup_{t_1 < t < t_2} y^+(t) < \epsilon_1$. The above argument also shows that $x^+(t) \geqslant k_3$ for any $t > t_1$. The proof is similar for the case $(x^+(t_1), y^+(t_1)) \in [0, \epsilon_1] \times [k_3, \infty)$. □

Lemma 3.6. *Let $(x^-(t,x,y), y^-(t,x,y))$ be the solution of the equation*

$$\dot{x}^-(t) = x^-(t)(a(-) - b(-)x^-(t) - c(-)y^-(t)),$$
$$\dot{y}^-(t) = y^-(t)(d(-) - e(-)x^-(t) - f(-)y^-(t)) \tag{3.15}$$

with $(x^-(0,x,y), y^-(0,x,y)) = (x,y)$. For any $\epsilon_1 > 0$ with $[k_3, \infty) \times [0, \epsilon_1] \subset \mathscr{U}$ (see (2.10)), we can find $\epsilon_2 > 0$ such that if $(x^-(t_1), y^-(t_1)) \in [k_3, \infty) \times [0, \epsilon_2]$ then $x^-(t) \geqslant k_3$ for any $t \geqslant t_1$ and $\sup_{t > t_1} y^-(t) \leqslant \epsilon_1$.

There is a similar result for the case $(x^-(t_1), y^-(t_1)) \in [0, \epsilon_2] \times [k_3, \infty)$.

Proof. The trick of finding ϵ_2 is similar as in the proof of Proposition 2.5 and we omit it here. □

Lemma 3.7. *For any $\delta > 0$, $\epsilon > 0$, there is $S_1 > 0$ such that if $\delta < x(t) < M$, $\epsilon \leqslant y(t) < M$ then $\inf_{0<s<S_1} x(t+s) \geqslant \delta/2$ and $\inf_{0<s<S_1} y(t+s) \geqslant \epsilon/2$.*

Proof. On the set $(0,M] \times (0,M]$, $\dot{x}/x$ and $\dot{y}/y$ are bounded below by constant, say γ, then $y(t+s) \geqslant y(t)e^{\gamma s}$. We can choose $0 < S_1 < |\ln 2/\gamma|$. □

Proposition 3.8. *Suppose that Conditions (3.10), (3.11) hold. Let $\omega(x, y)$ be the ω-limit set of the solution $(x(t,x,y), y(t,x,y))$ of (3.2) with $x > 0$, $y > 0$. Then, $(x^+, y^+) \in \omega(x, y)$.*

Proof. For convenience, we suppose $\xi_0 = +$. Let

$$0 < 2\varepsilon = \min\left\{\frac{\lambda}{pb(+)+qb(-)}, \frac{\theta}{pf(+)+qf(-)}\right\}, \quad \epsilon_3 = \min\{\epsilon_1, \epsilon_2\},$$

where ϵ_1 mentioned in Lemma 3.5 is chosen such that $\epsilon > \epsilon_1$ and $\epsilon_2 = \epsilon_2(\epsilon_1)$ is given in Lemma 3.6. We set

$$x_n = x(\tau_n, x, y); \quad y_n = y(\tau_n, x, y), \quad \mathscr{F}_0^n = \sigma(\tau_k : k \leqslant n); \quad \mathscr{F}_n^\infty = \sigma(\tau_k - \tau_n : k > n). \tag{3.16}$$

We see that (x_n, y_n) is $\mathscr{F}_0^n$-adapted. Moreover, if ξ_0 is given then $\mathscr{F}_0^n$ is independent of $\mathscr{F}_n^\infty$. We construct a sequence

$$\eta_1 = \inf\{2k+1 : x_{2k+1} \geqslant k_3; y_{2k+1} \geqslant \epsilon_3 \text{ or } x_{2k+1} \geqslant \epsilon_3; y_{2k+1} \geqslant k_3\},$$

$$\eta_2 = \inf\{2k+1 > \eta_1 : x_{2k+1} \geqslant k_3; y_{2k+1} \geqslant \epsilon_3 \text{ or } x_{2k+1} \geqslant \epsilon_3; y_{2k+1} \geqslant k_3\}$$

$$\dots$$

$$\eta_n = \inf\{2k+1 > \eta_{n-1} : x_{2k+1} \geqslant k_3; y_{2k+1} \geqslant \epsilon_3 \text{ or } x_{2k+1} \geqslant \epsilon_3; y_{2k+1} \geqslant k_3\}$$

$$\dots$$

$\eta_1 < \eta_2 < \cdots < \eta_k < \cdots$ is a sequence of $\mathscr{F}_0^n$-stopping times (see [7]). Moreover, $\{\eta_k = n\} \in \mathscr{F}_0^n$ for any k, n. Thus the event $\{\eta_k = n\}$ is independent of $\mathscr{F}_n^\infty$.

We show that $\eta_n < \infty$ a.s. for any n. Suppose to the contrary that there is N such that the set $\Gamma = \{\omega : \eta_N = \infty; \eta_{N-1} < \infty\}$ has a positive probability. Since either $x_{2k+1} \geqslant k_3$ or $y_{2k+1} \geqslant k_3$ so if $\omega \in \Gamma$ then either $x_{2k+1}(\omega) \geqslant k_3$; $y_{2k+1}(\omega) < \epsilon_3$ or $x_{2k+1}(\omega) < \epsilon_3$; $y_{2k+1}(\omega) \geqslant k_3$ for any $2k+1 > \eta_{N-1}(\omega)$. Let $2k+1 > \eta_{N-1}$, by virtue of Lemma 3.6, if $x_{2k+1} \geqslant k_3$; $y_{2k+1} < \epsilon_3$ then $x_{2k+2} \geqslant k_3$; $y_{2k+2} < \epsilon_1$. Hence, by Lemma 3.5, it follows that $x_{2k+3} \geqslant k_3$ which implies that $y_{2k+3} < \epsilon_3$. Thus, if $x_{2k+1} \geqslant k_3$; $y_{2k+1} < \epsilon_3$ then $x_n \geqslant k_3$; $y_n < \epsilon_1$ for any $n > 2k+1$. Using once more Lemmas 3.5 and 3.6 we get $\sup_{t > \eta_{N-1}} y(t) \leqslant \epsilon_1$. This contradicts with $\limsup_{t\to\infty} y(t) \geqslant 2\epsilon > \epsilon_1$.

On the other hand, let $A_k = \{\sigma_{\eta_k+1} < s, \sigma_{\eta_k+2} > t\}$ then

$$\mathbf{P}(A_k) = \mathbf{P}\{\sigma_{\eta_k+1} < s, \sigma_{\eta_k+2} > t\}$$

$$= \sum_{n=0}^{\infty} \mathbf{P}\{\sigma_{\eta_k+1} < s, \sigma_{\eta_k+2} > t | \eta_k = 2n+1\}\mathbf{P}\{\eta_k = 2n+1\}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \mathbf{P}\{\sigma_{2n+2} < s, \sigma_{2n+3} > t \mid \eta_k = 2n+1\}\mathbf{P}\{\eta_k = 2n+1\} \\
&= \sum_{n=0}^{\infty} \mathbf{P}\{\sigma_{2n+2} < s, \sigma_{2n+3} > t\}\mathbf{P}\{\eta_k = 2n+1\} \\
&= \sum_{n=0}^{\infty} \mathbf{P}\{\sigma_2 < s, \sigma_3 > t\}\mathbf{P}\{\eta_k = 2n+1\} = \mathbf{P}\{\sigma_2 < s, \sigma_3 > t\} > 0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathbf{P}(A_k \cap A_{k+1}) &= \mathbf{P}\{\sigma_{\eta_k+1} < s, \sigma_{\eta_k+2} > t, \sigma_{\eta_{k+1}+1} < s, \sigma_{\eta_{k+1}+2} > t\} \\
&= \sum_{0 \leqslant l < n < \infty} \mathbf{P}\{\sigma_{\eta_k+1} < s, \sigma_{\eta_k+2} > t, \sigma_{\eta_{k+1}+1} < s, \sigma_{\eta_{k+1}+2} > t \mid \\
&\quad \eta_k = 2l+1, \eta_{k+1} = 2n+1\}\mathbf{P}\{\eta_k = 2l+1, \eta_{k+1} = 2n+1\} \\
&= \sum_{0 \leqslant l < n < \infty} \mathbf{P}\{\sigma_{2l+2} < s, \sigma_{2l+3} > t, \sigma_{2n+2} < s, \sigma_{2n+3} > t \mid \\
&\quad \eta_k = 2l+1, \eta_{k+1} = 2n+1\}\mathbf{P}\{\eta_k = 2l+1, \eta_{k+1} = 2n+1\} \\
&= \sum_{0 \leqslant l < n < \infty} \mathbf{P}\{\sigma_{2n+2} < s, \sigma_{2n+3} > t\}\mathbf{P}\{\sigma_{2l+2} < s, \sigma_{2l+3} > t \mid \\
&\quad \eta_k = 2l+1, \eta_{k+1} = 2n+1\}\mathbf{P}\{\eta_k = 2l+1, \eta_{k+1} = 2n+1\} \\
&= \sum_{0 \leqslant l < n < \infty} \mathbf{P}\{\sigma_2 < s, \sigma_3 > t\}\mathbf{P}\{\sigma_{2l+2} < s, \sigma_{2l+3} > t \mid \\
&\quad \eta_k = 2l+1, \eta_{k+1} = 2n+1\}\mathbf{P}\{\eta_k = 2l+1, \eta_{k+1} = 2n+1\} \\
&= \mathbf{P}\{\sigma_2 < s, \sigma_3 > t\} \sum_{0 \leqslant l < n < \infty} \mathbf{P}\{\sigma_{2l+2} < s, \sigma_{2l+3} > t \mid \\
&\quad \eta_k = 2l+1, \eta_{k+1} = 2n+1\}\mathbf{P}\{\eta_k = 2l+1, \eta_{k+1} = 2n+1\} \\
&= \mathbf{P}\{\sigma_2 < s, \sigma_3 > t\} \sum_{l=0}^{\infty} \mathbf{P}\{\sigma_{2l+2} < s, \sigma_{2l+3} > t \mid \eta_k = 2l+1\}\mathbf{P}\{\eta_k = 2l+1\} \\
&= \mathbf{P}\{\sigma_2 < s, \sigma_3 > t\}^2 \ldots
\end{aligned}
$$

Thus,

$$
\mathbf{P}(A_k \cup A_{k+1}) = 1 - (1 - \mathbf{P}\{\sigma_2 < s, \sigma_3 > t\})^2.
$$

Continuing this way we obtain

$$\mathbf{P}\left(\bigcup_{i=k}^{n} A_i\right) = 1 - (1 - \mathbf{P}\{\sigma_2 < s, \sigma_3 > t\})^{n-k+1}.$$

Hence,

$$\mathbf{P}\left(\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty} A_i\right) = P\{\omega : \sigma_{\eta_n+1} < s; \sigma_{\eta_n+2} > t \text{ i.o. of } n\} = 1. \tag{3.17}$$

Suppose that U_δ is any δ-neighborhood of (x^+, y^+). We choose S_1 to be one in Lemma 3.7 and $T_3^*(\epsilon_3/2, \delta)$ to be one in Lemma 3.4. From (3.17) we see that there are infinitely many n such that either $x_{2n+1} \geqslant k_3, y_{2n+1} \geqslant \epsilon_3$ or $x_{2n+1} \geqslant \epsilon_3, y_{2n+1} \geqslant k_3$ with $\sigma_{2n+2} < S_1$; $\sigma_{2n+3} > T_3^*$. Therefore, either $x_{2n+2} \geqslant k_3/2, y_{2n+2} \geqslant \epsilon_3/2$ or $x_{2n+2} \geqslant \epsilon_3/2, y_{2n+2} \geqslant k_3/2$ which implies $(x_{2n+3}, y_{2n+3}) \in U_\delta$ for infinite many n. This means that $(x^+, y^+) \in \omega(x, y)$. The proposition is proved. □

Proposition 3.9. *Suppose that Conditions* (3.10) *and* (3.11) *hold. Let* $\omega(x, y)$ *be the* ω*-limit set of the solution of* (3.2) $(x(t,x,y), y(t,x,y))$ *with* $x > 0$, $y > 0$. *Define set* $\mathscr{A} \subset \mathbb{R}_+ \times \mathbb{R}_+$ *such that* $\lim_{t\to\infty} y(t,x,y) = 0$ *and* $\mathscr{B} \subset \mathbb{R}_+ \times \mathbb{R}_+$ *such that* $\lim_{t\to\infty} x(t,x,y) = 0$.

(a) *The positive orbit* γ^- *of the solution* $(x^-(t,x^+,y^+), y^-(t,x^+,y^+))$ *of the equation* (3.15) *is a subset of* $\omega(x, y)$.
(b) *If* $(x^+, y^+) \in \mathscr{A}$ *then the interval* $[(a(-)/b(-), 0); (a(+)/b(+), 0)] \subset \omega(x, y)$.
(c) *If* $(x^+, y^+) \in \mathscr{B}$ *then the interval* $[(0, d(-)/f(-)); (0, d(+)/f(+))] \subset \omega(x, y)$.
(d) *If* $(x^+, y^+) \in \ell$ *then the part of* ℓ *linking* (x^+, y^+) *and* (x^-, y^-) *belongs to* $\omega(x, y)$. *Moreover, the positive orbit* γ^+ *of the solution* $(x^+(t,x^-,y^-), y^+(t,x^-,y^-))$ *of* (3.14) *is a subset of* $\omega(x, y)$. *In addition, if* $\gamma^+ \cap \mathscr{A} \neq \emptyset$ *then* $[(a(-)/b(-), 0); (a(+)/b(+), 0)] \subset \omega(x, y)$; *if* $\gamma^+ \cap \mathscr{B} \neq \emptyset$ *then* $[(0, d(-)/f(-)); (0, d(+)/f(+))] \subset \omega(x, y)$; *if* $\gamma^+ \subset \ell$ *then* $\omega(x, y)$ *is the part of* ℓ *linking* (x^+, y^+) *and* (x^-, y^-).

Proof. (a) We prove that $\gamma^- \subset \omega(x, y)$. Let $(x^*, y^*) \in \gamma^-$, i.e., there is $t^* > 0$ such that $(x^-(t^*,x^+,y^+), y^-(t^*,x^+,y^+)) = (x^*, y^*)$. By the continuity of the solution in initial conditions, for any neighborhood V_ϵ of (x^*, y^*), there are $t_1 < t^* < t_2$ and $\delta > 0$ such that if $(u, v) \in U_\delta(x^+, y^+)$ then $(x^-(t,u,v), y^-(t,u,v)) \in V_\epsilon(x^*, y^*)$ for any $t_1 < t < t_2$.

Let

$$\zeta_1 = \inf\{2k+1 : (x_{2k+1}, y_{2k+1}) \in U_\delta(x^+, y^+)\},$$

$$\zeta_2 = \inf\{2k+1 > \zeta_1 : (x_{2k+1}, y_{2k+1}) \in U_\delta(x^+, y^+)\}$$

$$\dots$$

$$\zeta_n = \inf\{2k+1 > \zeta_{n-1} : (x_{2k+1}, y_{2k+1}) \in U_\delta(x^+, y^+)\}$$

$$\dots$$

By Proposition 3.8 we have $\zeta_k < \infty$ and $\lim_{k\to\infty} \zeta_k = \infty$ a.s.. Since $\{\zeta_k = n\} \in \mathscr{F}_0^n$ then $\{\zeta_k = n\}$ is independent of $\mathscr{F}_n^\infty$. Therefore,

$$\mathbf{P}\{\sigma_{\zeta_k+1} \in (t_1, t_2)\} = \sum_{n=0}^{\infty} \mathbf{P}\{\sigma_{\zeta_k+1} \in (t_1, t_2) | \zeta_k = 2n+1\}\mathbf{P}\{\zeta_k = 2n+1\}$$

$$= \sum_{n=0}^{\infty} \mathbf{P}\{\sigma_{2n+2} \in (t_1, t_2) | \zeta_k = 2n+1\}\mathbf{P}\{\zeta_k = 2n+1\}$$

$$= \sum_{n=0}^{\infty} \mathbf{P}\{\sigma_{2n+2} \in (t_1, t_2)\}\mathbf{P}\{\zeta_k = 2n+1\}$$

$$= \sum_{n=0}^{\infty} \mathbf{P}\{\sigma_2 \in (t_1, t_2)\}\mathbf{P}\{\zeta_k = 2n+1\} = \mathbf{P}\{\sigma_2 \in (t_1, t_2)\}.$$

Similarly,

$$\mathbf{P}\{\sigma_{\zeta_k+1} \in (t_1, t_2), \sigma_{\zeta_{k+1}+1} \in (t_1, t_2)\} = \mathbf{P}\{\sigma_2 \in (t_1, t_2)\}^2 \ldots,$$

which implies that

$$\mathbf{P}\{\omega : \sigma_{\zeta_n+1} \in (t_1, t_2) \ i.o. \text{ of } n\} = 1.$$

Since $(x_{\zeta_k}, y_{\zeta_k}) \in U_\delta(x^+, y^+)$ and $\sigma_{\zeta_k+1} \in (t_1, t_2)$ then $(x_{\zeta_k+1}, y_{\zeta_k+1}) \in V_\epsilon(x^*, y^*)$ for infinite many k. This means that $(x^*, y^*) \in \omega(x, y)$.

(b) Since $\gamma^- \subset \omega(x, y)$ and $(x^+, y^+) \in \mathscr{A}$, $(a(-)/b(-), 0)$ is in the closure $\bar{\gamma}^-$ of γ^-, which implies $(a(-)/b(-), 0) \in \omega(x, y)$. Taking $(x^*, y^*) \in [(a(+)/b(+), 0); (a(-)/b(-), 0)]$ and an arbitrary V_ϵ-neighborhood of (x^*, y^*), we see that there are $t_1 < t_2$ and $\delta > 0$ such that if $(x_1, y_1) \in U_\delta(a(-)/b(-), 0)$ then $(x^+(t, x_1, y_1), y^+(t, x_1, y_1)) \in V_\epsilon(x^*, y^*)$ for any $t_1 < t < t_2$. Denote $\rho_0 = 0, \rho_k = \inf\{2k > \rho_{k-1} : (x_{2k}, y_{2k}) \in U_\delta\}$. By a similar trick as above, we can prove that (x_{2k+1}, y_{2k+1}) visits the neighborhood $V_\epsilon(x^*, y^*)$ at infinitely many times.

(c) The proof is similar as (b).

(d) Since ℓ is the stable manifold of system (3.15) then if $(x^+, y^+) \in \ell$ we have $\gamma^- = \gamma^-(x^+, y^+) \subset \ell$. Moreover, $\lim_{t\to\infty}(x^-(t, x^+, y^+), y^-(t, x^+, y^+)) = (x^-, y^-)$. Thus, by the continuity of the solution in the initial conditions, it follows that the part of ℓ linking (x^+, y^+) and (x^-, y^-) is a subset of $\omega(x, y)$.

By noticing that $(x^-, y^-) \in \overline{\gamma^-}$ and $\omega(x, y)$ is a closed set then $(x^-, y^-) \in \omega(x, y)$. Therefore by a similar argument we conclude that the positive orbit γ^+ of the solution $(x^+(t, x^-, y^-), y^+(t, x^-, y^-))$ of (3.14) is a subset of $\omega(x, y)$

Further, if $(x^*, y^*) \in \gamma^+(x^-, y^-) \cap \mathscr{A}$ then for any neighborhood U_ϵ of (x^*, y^*), there is a V_δ-neighborhood of (x^-, y^-) and $t_1 < t_2$ such that $(x^+(t, x_1, y_1), y^+(t, x_1, y_1)) \in U_\epsilon$ for any $(x_1, y_1) \in V_\delta$ and $t_1 < t < t_2$. The argument is repeated as in the part (a).

Finally, if $\gamma^+(x^-, y^-) \subset \ell$ and $\gamma^-(x^+, y^+) \subset \ell$ we follow that $\gamma^+(x^-, y^-) = \gamma^-(x^+, y^+)$. By the continuity of the solution in the initial conditions and Proposition 3.8, the set $\gamma^+(x^-, y^-)$ is stable. Hence it yields the result. The proof is complete. □

We illustrate the above model by following numerical examples.

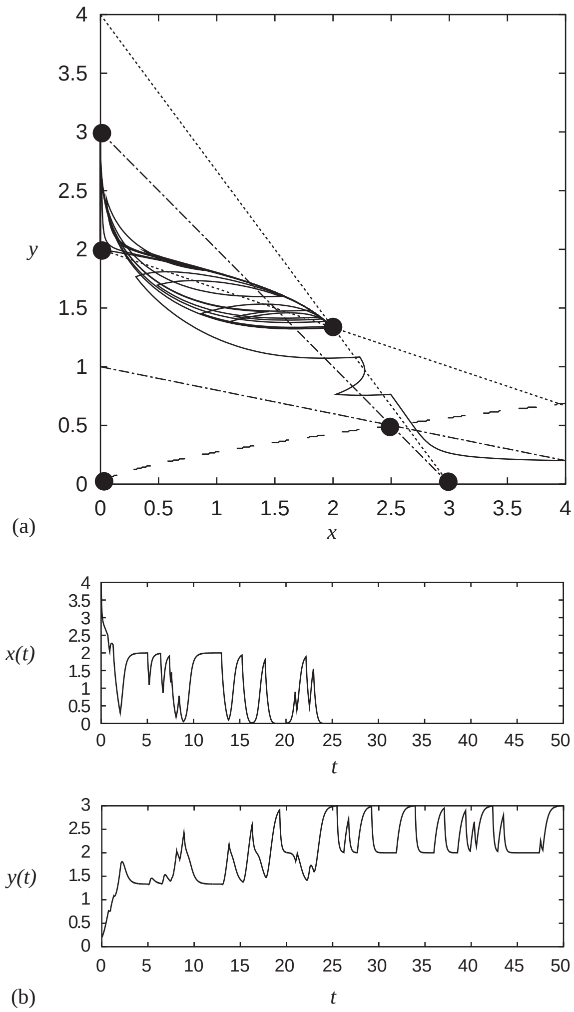

Fig. 2. Case I. Hypothesis (3.10) does not hold, that is, $\lambda < 0$. The parameters are $a(+) = 12$, $b(+) = 4$, $c(+) = 3$, $d(+) = 6$, $e(+) = 1$, $f(+) = 3$, $a(-) = 5$, $b(-) = 1$, $c(-) = 5$, $d(-) = 3$, $e(-) = 1$, $f(-) = 1$. The initial condition is $(x(0), y(0)) = (4, 0.2)$. (a) The x–y phase plane. The solid line is a solution of System (3.2). The dotted and dot-dashed lines are null clines of the systems (1.1) with constant coefficients corresponding to (+) and (−), respectively. Solid dots are equilibrium points of the two systems. The broken line indicates ℓ. (b) The temporal fluctuation of the solution.

Case I: Systems do not satisfy the hypotheses (3.10) and (3.11). The solution $(x(t,x,y), y(t,x,y))$ has a component tending to 0 (see Fig. 2).

Case II: The solution $(x(t,x,y), y(t,x,y))$ oscillates between the stable point (x^+, y^+) and the boundary point $(0, d(-)/f(-))$ (see Fig. 3).

Case III: Example of the system satisfying $(x^+, y^+) \in \ell$ and $\gamma^+ \cap \mathscr{B} \neq \emptyset$ but $\gamma^+ \cap \mathscr{A} = \emptyset$. The $\omega(x, y)$ includes the boundary interval $[(0, d(-)/f(-)); (0, d(+)/f(+))]$ (see Fig. 4).

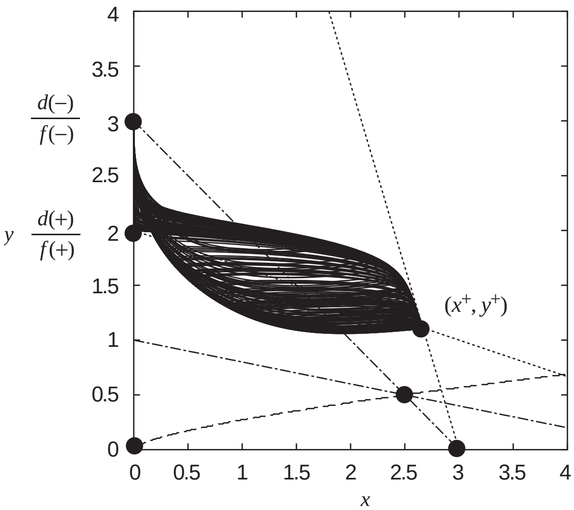

Fig. 3. Case II. The solution of System (3.2) with the initial condition $(x(0), y(0)) = (4, 0.2)$ is plotted for $t \in [500, 1000]$. The parameters are $a(+) = 30$, $b(+) = 10$, $c(+) = 3$, $d(+) = 6$, $e(+) = 1$, $f(+) = 3$, $a(-) = 5$, $b(-) = 1$, $c(-) = 5$, $d(-) = 3$, $e(-) = 1$, $f(-) = 1$. The explanations for the lines and dots are given in Fig. 2.

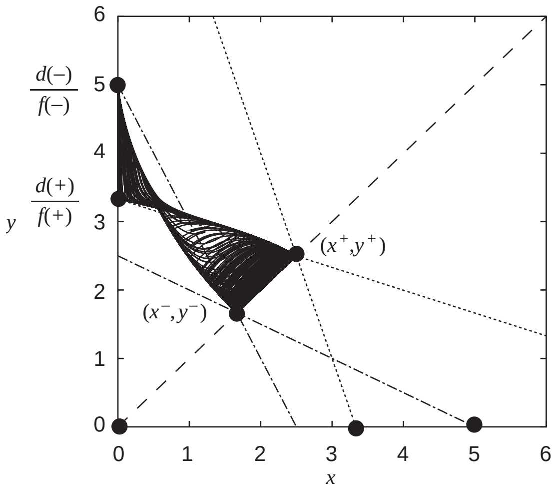

Fig. 4. Case III. The solution of System (3.2) with the initial condition $(x(0), y(0)) = (4, 0.2)$ is ploted for $t \in [500, 2000]$. The parameters are $a(+) = 20$, $b(+) = 6$, $c(+) = 2$, $d(+) = 30$, $e(+) = 3$, $f(+) = 9$, $a(-) = 10$, $b(-) = 2$, $c(-) = 4$, $d(-) = 10$, $e(-) = 4$, $f(-) = 2$. The explanations for the lines and dots are given in Fig. 2.

Case IV: Example of the system satisfying $(x^+, y^+) \in \ell$, $\gamma^+ \cap \mathscr{A} \neq \emptyset$ and $\gamma^+ \cap \mathscr{B} \neq \emptyset$. The $\omega(x, y)$ includes the boundary intervals $[(0, d(-)/f(-)); (0, d(+)/f(+))]$ and $[(a(-)/b(-), 0); (a(+)/b(+), 0)]$ (see Fig. 5).

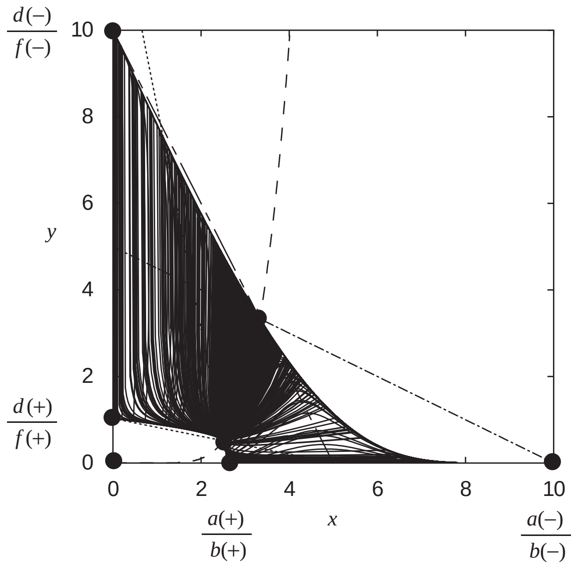

Fig. 5. Case IV. The solution of System (3.2) with the initial condition $(x(0), y(0)) = (4, 0.2)$ is plotted for $t \in [10000, 20000]$. The parameters are $a(+) = 13.39$, $b(+) = 5.15$, $c(+) = 1$, $d(+) = 18.54$, $e(+) = 3.708$, $f(+) = 18$, $a(-) = 2$, $b(-) = 0.2$, $c(-) = 0.4$, $d(-) = 19.84$, $e(-) = 3.968$, $f(-) = 1.984$. The explanations for the lines and dots are given in Fig. 2.

4. Discussion

In this paper, we study the trajectory behavior of a Lotka–Volterra competition system. In the first part the non-autonomous systems satisfying the bistable condition are analyzed, and it is shown that there exists a unique solution, bounded above and below by positive constants. In the next we introduce telegraph noises which result in the parameter switching between the stable system and the bistable. In that case we observe the oscillatory behavior of the solution.

From the viewpoint of biological modeling, we give an answer to the destiny of the competitive populations under the temporally variable environment.

Especially we assume that the environmental change directly affects the model parameters. It may be probable that the environmental fluctuation is not so large that the qualitative character of the model does not alter. Then we consider it with deterministic variable under the bistable conditions in the first part.

More interestingly, what happens if the external environment greatly changes such that the spontaneous switching between favorable and unfavorable conditions frequently occurs, i.e., one species can persist during some periods but should go extinct during another periods. In reality, for example, we can observe such distinctive seasonal change as dry and wet seasons in monsoon forest, and it may strongly affect the characteristics of the vegetation in the forest. Also in boreal and arctic regions, seasonality exerts a strong influence on the dynamics of mammals, and [9] analyze the model including the effect of seasonality by deterministically spontaneous switching of parameters and equations corresponding to Fennoscandian summer and winter.

Here we restrict to analyze the model with the exponentially distributed spontaneous switching between only two environmental states, and show the complex behavior in the transient due to stochastic dynamics and the existence of the oscillatory attractor in the limit. It tells us the possibility of oscillation of competitive populations caused by the environmental large fluctuation which alters the habitat qualitatively, even when the environmental stochasticity is included.

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