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Conflicting chip firing games on graphs and on trees

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Abstract. Chip Firing Games on (directed) graph are widely used in theoretical computer science and many other sciences. In this model, chips are fired from one vertex to all of its neighbors at the same time. The purpose of our paper is to study an extended version of this model, the Conflicting Chip Firing Game, by considering that chips can be fired from one vertex to one of its neighbors at each time. Our main results are obtained when the support graph of this game is a rooted tree. In this case, we give the characterization of its reachable configurations and of its fixed points. Moreover we show the local lattice structure of its configuration space.

Keywords: Chip Firing Game, conflicting game, convergence, discrete dynamical system, evolution rule, fixed point, tree.

1. Introduction

A Chip Firing Game (CFG) [1,2] is defined on a directed (multi) graph as follows. A configuration of the game is a distribution of chips on the vertices of the graph, and the evolution rule (firing a vertex) is defined by: a configuration can be transformed into another one by transferringa chipfrom one vertex along each of its outgoing edges, if it contains at least as many chips as its outgoing degree. The set of all configurations reachable from the initial one is called *configuration space*, and a *fixed point* is a configuration from which the evolution rule can not be applied. Convergence conditions (involving the number of chips or the structure of the graph) are given in [1-3] as well as different proofs of the fact that the configuration space of any convergent CFG is a lattice. See Figure 1 for an example.

CFGs are widely used in theoretical computer science, in physics and in economics. For example, CFGs model distributed behaviors (such as dynamical distribution of jobs over a network [4,5]), combinatorial objects (such as integer partitions [6-9], dollar game [10,11] and other [12]). In physics, it is mainly studied as a paradigm for the so called *Self Organized Criticality*, an important area of research [13-15]. It is also proved in [16] that infinite CFGs are Turing complete, which shows the potential complexity of their behaviors.

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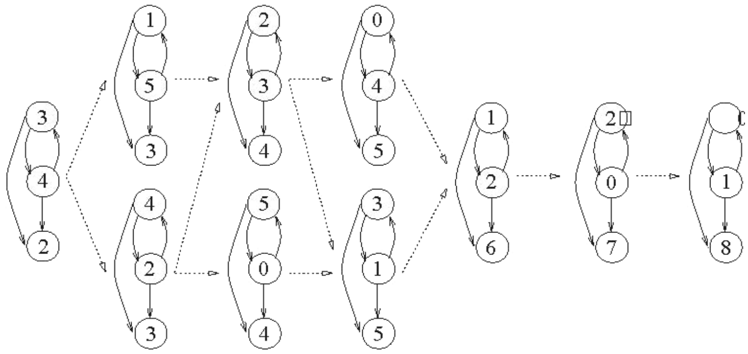

Fig. 1. The configuration space of a CFG with 9 chips. The weight of each vertex is indicated, and the shaded vertices are the ones which can be fired.

We observe that in CFGs, the condition for firing a vertex is quite strict: this vertex must contain at least as many chips as its outgoing degree. However, in many model, for example models in distributed systems [5] or in economics [11], chips can be fired from a vertex to one of its neighbors if this vertex has at least one chip. And in this case, chips are not transferring to all neighbors of this vertex at the same time, but at different times. In order to modelize these systems, we investigate an extended version of CFG, by considering that a configuration can be transformed into another one by transferring chips from one vertex along one of its outgoing edges. However, the firing of a chip along one edge may cause a conflict with the one along another edge. Hence we call our model "Conflicting Chip Firing Game" (CCFG).

Further, we constate that, in this new model, by relaxing the condition about the number of chips in a vertex, the evolution rule is much more flexible. In other side, the obtained configuration space has not the lattice structure, and the convergence properties. This situation is illustrated at the end of Section 2. Moreover, we note that it is more difficult to find a support graph which has good properties in CCFG model than in CFG model.

In Section 3, we consider a particular but important case of CCFGs, where the support graph is a rooted tree. We characterize the reachable configurations and fixed points of the model. At the end, we study the complexity as well as the local lattice structure of the configuration space.

Before entering in the core of this paper, letus give here some preliminary notations of order and lattice theory. A binary relation $\leq$ over a set P is said to be an order if it is reflexive, transitive and anti-symmetric. The set P together with the relation $\leq$ is then called a partially ordered set, or simply a *poset*. A poset L is a *lattice* if any two elements x and y of L have a greatest lower bound, called the infimum of x and y and denoted by *inf*(x, y), anda smallest greater bound, called the supremum of x and y and denoted by *sup*(x, y). The study of lattices is an important part of order theory, and many results about them exist. In particular, various classes of lattices have been defined and appear in computer science, mathematics, physics, social sciences, and others. For more details about orders and lattices, we refer to [17].

2. The general model conflicting chip firing games

In this section after giving the definition of the model Conflicting Chip Firing Games (*CCFG*), we investigate the relation between its configurations.

Let $G = (V, E, c)$ be a (weighted) directed graph where $V = \{1,2,...,m\}$ being the set of vertices of G, $E = \{t_1,t_2,...,t_p\} \subset V \times V$ being the set of edges of G, and the *capacity function* c being a function from E to N. A *CCFG* on $G = (V, E, c)$ (G is called the *support* or the *base* of the game) with n chips is defined as follows. A configuration $a = (a_1, a_2,..., a_m)$ of the game is a distribution of n chips into V, where the *weight* a_i associated with each vertex i can be regarded as the number of chips stored at the vertex i. The evolution rule, called also *transition rule* or *firing rule* is the following: an edge (i, j) can be fired if the vertex i contains at least $c(i, j)$ chips, and the firing of this edge is the transferring of $c(i, j)$ chips from vertex i to vertex j. Moreover, a *firing sequence* is a sequence of firings.

Let G be a support graph, and let O be a configuration, we call *configuration space*, and we denote by $CCFG(G,O)$, the set of all configurations reachable from the initial configuration O. On this set, we define the following relation: $a \geq b$ if b can be obtained from a by applying a sequence of firings.

In order to describe the evolution of a *CCFG* on graph G, we introduce the *evolution matrix M(G)* as follows: $M(G) = (a_{qi})_{p \times m}$ where $a_{qi} = c(t_q)$ (resp. $a_{qi}= -c(t_q)$) if i is the going out (resp. going in) vertex of edge t_q, and $a_{qi} = 0$ otherwise. Denote by $e[q]$ the unit p-parts vector, where the position q is equal to 1, and the others are equal to 0. We constate that if configuration b is obtained from configuration a by firing edge t_q then $b = a+ e[q] \times M$. Further, let $C = (t_{i_1},...,t_{i_r})$ be a firing sequence from a to b where the edges $t_{i_1},...,t_{i_q}$ are subsequently fired. We define the *shot vector* of C being the vector $k(C) = (k_1,...,k_p)$ where k_q is the number of occurrences of t_q in C. The following result is direct from the above definitions:

Proposition 1: Let C be a firing sequence from a to b, then we have: $b= a+ k(C) \times M(G)$.

Let us give in Figure 2 a small example of this game. From this figure, we see that configuration (1, 2, 6) is obtained from (3, 4, 2) by the sequence $C = t_1t_2t_3t_4t_1t_2t_3t_4$. So the shot vector of C is (2, 2, 2, 2).

Moreover, from this small example, we can observe that in constrat with the case of the classical *CFG*, a *CCFG* may have cycles (firing sequences come from a configuration and come back to itself) and have many fixed points. Therefore, it has not a lattice structure. However, in some cases where the support graph has some "good" properties, this structure is maintained. In the next section, we study such a particular (and important) class of *CCFG*.

3. *CCFG* on a tree

The purpose of this section is to investigate a class of *CCFG* whose the support graph is a rooted tree with edges directed from nodes to their children. We show a characterization for reachable configurations and for fixed points of this game. This allows us to describe the complexity of the game by giving the cardinality of its configuration space. We also prove the local lattice structure of this space.

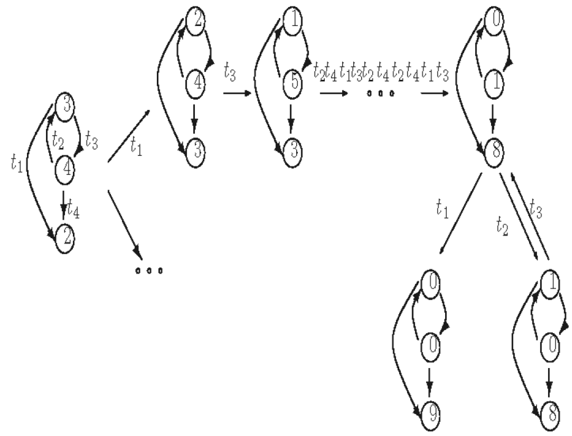

Fig. 2. Some configurations of a CCFG with 9 chips.

First of all, we present here some preliminary definitions.

Definition 1: Let $T = (V, E)$ be a rooted tree with $V = \{1, ..., n\}$, a *node* is a vertex of T, a *leaf* is a node having no child, and the *depth* of a node v, denoted by $d(v)$, is the length of the unique path from the root to v.

Definition 2: Let n be a positive integer and let S be a set with $|S| = k$. A *composition* of n into S is an ordered sequence $(a_1, a_2, ..., a_k)$ of non negative integers such that $a_1 + a_2 + ... + a_k = n$. The integer a_i is called the *weight* of i.

Next, we define for each composition a of n into V, the *horizontal energy* as follows:

Definition 3: Let $T = (V, E)$ be a rooted tree and let $a = (a_1, ..., a_m)$ be a composition of n into V. The *horizontal energy* $e_H(i, a)$ at node i of a is the quality $a_i d(i)$. And the *horizontal energy* of a is the quality $e_H(a) = \sum_{i \in V} e_H(i, a)$.

Now, the *CCFG* on T with n chips, denoted by *CCFG(T,n)*, is defined as follows:

- Each configuration is a composition of n into V;
- In the initial configuration $\mathcal{O}$, all n chips are centered at the root, and there is no chip at other nodes;
- Evolution rule: the node i can give one chip to the node j, one of its children, if i has at least one chip.

We denote also the configuration space of this game by $CCFG(T, n)$, and we write $b \leq a$ if b can be obtained from a by a firing sequence. In particular, we write $a \rightarrow b$ if b is obtained from a by applying once evolution rule. It is clear that $e_H(b) = e_H(a) - 1$. This implies the following result.

Lemma 1: The configuration space *CCFG(T, n)* has no cycle and consequently it is stationary. Moreover, the set *CCFG(T, n)* equipped with the relation ≤ is a poset.

Figure 3 shows an example of a *CCFG* on a tree of 5 nodes with 2 chips.

In the next propositions, we give a characterization of configurations of *CCFG(T,n)* as well as its fixed points.

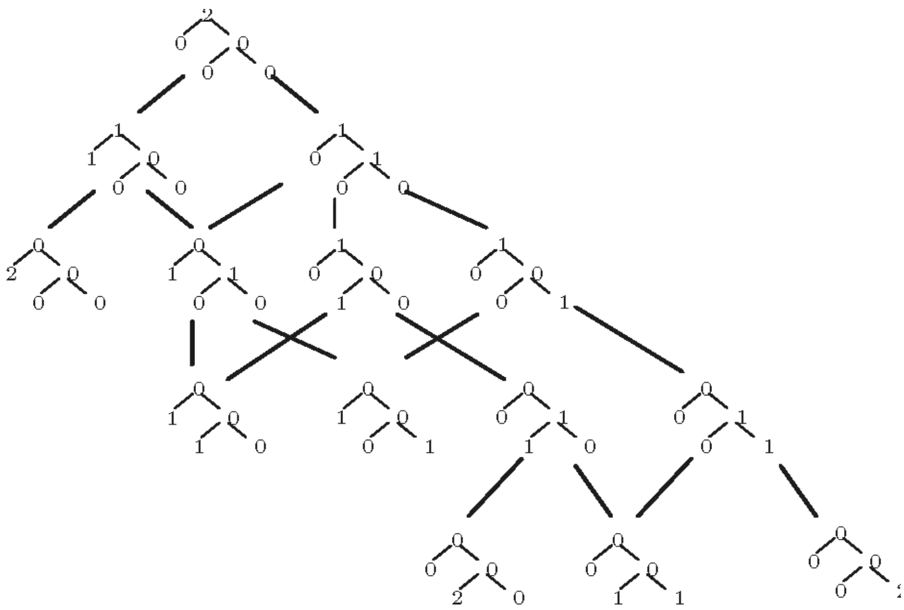

Fig. 3. The configuration space of a CCFG on tree.

Proposition 2: The set *CCFG(T,n)* is exactly the set of compositions of n into V. Consequently, $CCFG(T,n)$ has exactly $\binom{n+m-1}{m-1}$ configurations.

Proof: Let $a = (a_1,a_2,\ldots,a_m)$ be a composition of n into V. It is clear that if $e_H(a) = 0$ then a have no chips at any node but the root of T, that means a is nothing but $\mathcal{O}$. In the case $e_H(a) > 0$, we prove that there exists a firing sequence from the initial configuration $\mathcal{O}$ to a. For that, it is sufficient to show that there exists a composition a' of n into V such that $a' \to a$ and $e_H(a') < e_H(a)$. Because $e_H(a) > 0$, there exists a node i such that $a_i > 0$. Let j be the father of i. We consider the composition a' obtained form a by increasing a_j by 1 and by decreasing a_i by 1. It is easy to check that a' is a composition of n into V satisfying $a' \to a$ and $e_H(a') = e_H(a) - 1$.

This result implies that the number of configurations of $CCFG(T,n)$ is the number of non-negative integer solutions of the equation $x_1 + x_2 + \ldots + x_m = n$. Hence it is $\binom{n+m-1}{m-1}$. The proof is completed.

Then the following result is straightforward:

Corollary 1: The fixed points of $CCFG(T,n)$ are compositions of n into the set of leaves of T. Consequently, the number of fixed points of $CCFG\ (T,n)$ is $\binom{n+l-1}{l-1}$, where l is the number of leaves of T.

In the previous section, while studying the general model *CCFG*, we show a necessary condition by shot vector for two configurations to be comparable. However, we have not yet given any sufficient condition for this. In constrat, withthe support graph beinga tree, we can describe explicitely the order between configurations by introducing the following notation of vertical energy.

Definition 4: Let $T = (V, E)$ be a rooted tree and let $a = (a_1,\ldots,a_m)$ be a composition of n into V. The *vertical energy* $e_V(i,a)$ at node i of a is equal to the number of chips in the subtree of T

rooted at i. And the *vertical energy* of a is the quality $e_V(a) = \sum_{i \in V} e_V(i,a)$.

We observe that $e_V(a) = e_H(a)$. Moreover, if the node i has children $i_1, i_2, \ldots, i_k$ then $e_V(i,a) = e_V(i_1, a) + e_V(i_2, a) + \ldots + e_V(i_k, a) + a_i$. Therefore, each configuration is determined uniquely by their vertical energies at all nodes of the support tree.

We can now state our result on the order of *CCFG* on tree:

Theorem 1: Let a and b be two configurations of *CCFG*(T, n). Then $a \geq b$ in *CCFG*(T, n) if and only if $e_V(i, a) \leq e_V(i, b)$ for all nodes i of T.

Proof: First, we prove the necessary condition. It is sufficient to prove the statement for the case $a \rightarrow b$. Assume that b is obtained a by transferring one chip from node i to j. Then $a_l = b_l$ for all $l \neq i, j$. Let k be a node of T. If $k \neq j$, then the subtree of T rooted at k contains either both i, j or none of them. So $e_V(k, a) = e_V(k, b)$. If $k = j$, then $e_V(j, a) = e_V(j, b) - 1$. Therefore, $e_V(k, a) \leq e_V(k, b)$ for all nodes k of T.

Conversely, we prove the sufficient condition for showing that there exists a firing sequence from a to b. It is clear that if $e_V(i, a) = e_V(i, b)$ for all nodes i then $a = b$. For other cases, we remark that there exists a node i such that $e_V(i, a) < e_V(i, b)$ and $e_V(k, a) = e_V(k, b)$ for all nodes tk of the subtree rooted at t. This implies that $a_i < b_i$. Let j be the father of i. Let c be the composition of n into V obtained from b by increasing b_j by 1 and by decreasing b_i by 1. It is easy to see that $c \rightarrow b$. So, by using the necessary condition, we have $e_V(i, c) = e_V(i, b) - 1$ and $e_V(k, c) = e_V(k, b)$ for all nodes $k \neq i$. Hence, $e_V(k, c) \geq e_V(k, a)$ for all nodes k of T. So by recurrence we also obtain an inverse firing sequence from b back to a. This completes the proof.

To finish this section, we investigate the structure of *CCFG* on tree. Let us recall that, in the classical model *CFG*, the configuration space has a lattice structure with a unique fixed point. Unfortunately, in the general *CCFG*, there are many fixed points in the configuration space and the structure of this space is quite complicate. Nervertheless, in the case the support of a *CCFG* is a rooted tree, we can prove the local lattice structure. Let us first recall that for any two elements $a \geq b$ in a poset, the interval $[b,a]$ is the set of all elements c suchthat $a \leq c \leq b$.

Theorem 2: Let $a \geq b$ be two configurations of *CCFG*(T,n). Then the interval $[b,a]$ is a graded lattice.

Proof: Since the interval $[b,a]$ has a mimimal element b, to prove its lattice structure it is sufficient to prove that for any two elements $c,d \in [b,a]$, there exists *sup*(c,d) (see [17]). To find this supremum, we first compute its vertical energies as follows. Put $e_i = min\{e_V(i,c), e_V(i,d)\}$ for every node i of V. It is clear that $e_1 = n$. In addition, if i has children $i_1, i_2, \ldots, i_k$, *then*

$$
\begin{aligned}
& e_{i_1} + e_{i_2} + \ldots + e_{i_k} \\
& = \min\{e_V(i_1,c), e_V(i_1,d)\} + \ldots + \min\{e_V(i_k,c), e_V(ik,d)\} \\
& \leq \min\{e_V(i_1,c) + \ldots + e_V(i_k,c), e_V(i_1,d) + \ldots + e_V(i_k,d)\} \\
& \leq \min\{e_V(i,c), e_V(i,d)\} = e_i .
\end{aligned}
$$

Now, let us define the following sequence of non-negative integers: $g = (g_1, g_2, \ldots, g_m)$ where $g_i = g_i - (g_{i_1} + g_{i_2} + \ldots + g_{i_k})$. It is clear that this sequence is a composition of n into V, that means g is a configuration of T. Furthermore, g has vertical energies $e_1, e_2, \ldots, e_m$. So by using Theorem 1, we have $g \geq c$ and $g \geq d$. On the other hand, let h be a configuration of *CCFG*(T,n) satisfying $h \geq c,d$. We have $e_V(i, h) \leq e_V(i, c)$ and $e_V(i, h) \leq e_V(i, d)$ for all nodes i of T, this implies that $e_V(i, h) \leq$

$min\{e_V(i,c), e_V(i,d)\} = e_V(i,g)$. Hence $g \leq h$. This proves that g is the supremum of c and d.

Finally, we remark that the horizontal energy is reduced by exactly 1after once applying the evolution rule. So the lattice $[b,a]$ is graded.

The following corollary is straightforward from the graded propertyn of the lattice.

Corollary 2: In the $CCFG(T,n)$ we have:

i) The maximal length of a firing sequence from the initial configuration to one fixed point is $n.\max_{i \in T}\{d(i)\}$.

ii) The minimal length of a firing sequence from the initial configuration to one fixed point is $n.\min_{i \in T}\{d(i)\}$.

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