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LOCAL DIMENSION OF FRACTAL MEASURE ASSOCIATED WITH THE $(0,1,a)$ - PROBLEM: THE CASE $a = 6$

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Abstract. Let $X_1, X_2, ...$ be a sequence of independent, identically distributed(i.i.d) random variables each taking values $0, 1, a$ with equal probability $1/3$. Let μ be the probability measure induced by $S = \sum_{n=1}^{\infty} 3^{-n} X_n$. Let $\alpha(s)$ (resp.$\underline{\alpha}(s), \overline{\alpha}(s)$) denote the local dimension (resp. lower, upper local dimension) of $s \in$ supp μ, and let

$$\overline{\alpha} = \sup\{\overline{\alpha}(s) : s \in \text{supp } \mu\}; \underline{\alpha} = \inf\{\underline{\alpha}(s) : s \in \text{supp } \mu\}$$

$$\mathrm{E} = \{\alpha : \alpha(s) = \alpha \text{ for some } s \in \text{supp } \mu\}.$$

In the case $a = 3$, $E = [2/3, 1]$, see [6]. It was hoped that this result holds true with $a = 3k$, for any $k \in \mathbb{N}$. We prove that it is not the case. In fact, our result shows that for $k = 2 (a = 6), \overline{\alpha} = 1$, $\underline{\alpha} = 1 - \frac{\log(1+\sqrt{5}) - \log 2}{2 \log 3} \approx 0.78099$ and $E = [1 - \frac{\log(1+\sqrt{5}) - \log 2}{2 \log 3}, 1]$.

1. Introduction

Let $X_1, X_2, ...$ be a sequence of i.i.d random variables each taking values $a_1, a_2, ..., a_m$ with probability $p_1, p_2, ..., p_m$ respectively. Then the sum

$$S = \sum_{n=1}^{\infty} \rho^n X_n$$

is well defined for $0 < \rho < 1$. Let μ be the probability measure induced by S, i.e.,

$$\mu(A) = \text{Prob}\{\omega : S(\omega) \in A\}.$$

It is known that the measure μ is either purely singular or absolutely continuous. In 1996, Lagarias and Wang[8] showed that if m is a prime number, $p_1 = p_2 = ... = p_m = 1/m$ and $a_1, ..., a_m$ are integers then μ is absolutely if and only if $\{a_1, a_2, ..., a_m\}$ forms a complete system(mod$m$), i.e., $a_1 \equiv 0 \pmod{m}, a_2 \equiv 1 \pmod{m}, \ldots, a_m \equiv m - 1 \pmod{m}$.

An intriguing case when $m = 3, p_1 = p_2 = p_3 = \frac{1}{3}$ and $a_1 = 0, a_2 = 1, a_3 = 3$, known as the "$(0, 1, 3) - Problem$", is of great interest and has been investigated since the last decade.

Let us recall that for $s \in \text{supp}\ \mu$ the local dimention $\alpha(s)$ of μ at s is defined by

$$\alpha(s) = \lim_{h\to 0^+} \frac{\log \mu(B_h(s))}{\log h}, \tag{1}$$

provided that the limit exists, where $B_h(s)$ denotes the ball centered at s with radius h. If the limit (1) does not exist, we define the upper and lower local dimension, denoted $\overline{\alpha}(s)$ and $\underline{\alpha}(s)$, by taking the upper and lower limits respectively.

Observe that the local dimension is a function defined in the supp μ. Denote

$$\overline{\alpha} = \sup\{\overline{\alpha}(s) : s \in \text{supp}\ \mu\}\ ;\ \underline{\alpha} = \inf\{\underline{\alpha}(s) : s \in \text{supp}\ \mu\};$$

and

$$E = \{\alpha : \alpha(s) = \alpha \text{ for some } s \in \text{supp}\ \mu\}$$

be the attainable values of $\alpha(s)$, i.e., the range of α.

In [6], T. Hu, N. Nguyen and T. Wang have investigated the "$(0,1,3)$- Problem" and showed that $E = [2/3, 1]$. In this note we consider the following general problem.
Problem. Describle the local dimension for the $(0,1,a)$- problem, where $a \in \mathbb{N}$ is a natural number.

Note that the local dimension is an important characteristic of singular measures. For $a = 3k+2$ the measure μ is absolutely continuous, therefore we only need to consider the case $a = 3k$ or $a = 3k+1, k \in \mathbb{N}$. For $a = 3k$ it is conjectured that the local dimension is still the same as $a = 3$, it means that $E = [2/3, 1]$. Our aim in this note is to disprove this conjecture. In fact, our result is the following:

Main Theorem. For $a = 6$ we have $\overline{\alpha} = 1, \underline{\alpha} = 1 - \frac{\log(1+\sqrt{5})-\log 2}{2\log 3}$ and $E = [1 - \frac{\log(1+\sqrt{5})-\log 2}{2\log 3}, 1]$.
The proof of the Main Theorem will be given in Section 3. The next section we establish some auxiliary results used in the proof of the Main Theorem.

2. Auxiliary Results

Let $X_1, X_2, \dots$ be a sequence of i.i.d random variables each taking values $0, 1, 6$ with equal probability $1/3$. Let $S = \sum_{n=1}^{\infty} 3^{-n} X_n$, $S_n = \sum_{i=1}^{n} 3^{-i} X_i$ be the n-partial sum of S, and let μ, μ_n be the probability measures induced by S, S_n respectively. For any $s = \sum_{n=1}^{\infty} 3^{-n} x_n \in \text{supp}\ \mu, x_n \in$ D: $= \{0,1,6\}$, let $s_n = \sum_{i=1}^{n} 3^{-i} x_i$ be it's n-partial sum. It is easy to see that for any $s_n, s'_n \in \text{supp}\ \mu_n$, $|s_n - s'_n| = k3^{-n}$ for some $k \in \mathbb{N}$, and for any interval between two consecutive points in supp μ_n there exists at least one point in supp μ_{n+1}. Let

$$\langle s_n \rangle = \{(x_1, x_2, ..., x_n) \in D^n : \sum_{i=1}^{n} 3^{-i} x_i = s_n\}.$$

Then we have

$$\mu_n(s_n) = \#\langle s_n \rangle 3^{-n} \text{ for every } n, \tag{2}$$

where $\#A$ denotes the cardinality of set A.

Two sequences $(x_1, x_2, ..., x_n)$ and $(x'_1, x'_2, ..., x'_n)$ in D^n are said to be *equivalent*, denoted by $(x_1, x_2, \dots, x_n) \approx (x'_1, x'_2, \dots, x'_n)$ if $\sum_{i=1}^n 3^{-i} x_i = \sum_{i=1}^n 3^{-i} x'_i$. Then we have

2.1.Claim. Assume that $(x_1, x_2, \dots, x_n)$ and $(x'_1, x'_2, \dots, x'_n)$ in D^n. If $(x_1, x_2, \dots, x_n) \approx (x'_1, x'_2, \dots, x'_n)$ and $x_n > x'_n$ then $x_n = 6, x'_n = 0$.
Proof. Since $(x_1, x_2, \dots, x_n) \approx (x'_1, x'_2, \dots, x'_n)$, we have

$$3^{n-1}(x_1 - x'_1) + 3^{n-2}(x_2 - x'_2) + \dots + 3(x_{n-1} - x'_{n-1}) + x_n - x'_n = 0,$$

which implies $x_n - x'_n \equiv 0 \pmod 3$, and by virtue of $x_n > x'_n$ we have $x_n - x'_n = 6$. Hence $x_n = 6, x'_n = 0$. The claim is proved.

Consequece 1. *a) Let $s_{n+1} \in supp\ \mu_{n+1}$ and $s_{n+1} = s_n + \frac{1}{3^{n+1}}, s_n \in supp\ \mu_n$. We have*

$$\#\langle s_{n+1}\rangle = \#\langle s_n\rangle \text{ for evrery } n.$$

b) For any $s_n, s'_n \in supp\ \mu_n$ such that $s_n - s'_n = \frac{1}{3^n}$, we have

$$\#\langle s_n\rangle \leqslant \#\langle s'_n\rangle.$$

Proof. Observe that a) is a directive consequence of Claim 2.1.

b) It is easy to see that if $s_n - s'_n = \frac{1}{3^n}$, then $s_n = s_{n-1} + \frac{1}{3^n}$ and $s'_n = s_{n-1} + \frac{0}{3^n}$, where $s_{n-1} \in \text{supp}\ \mu_{n-1}$. Therefore from a) it follows that

$$\#\langle s_n\rangle = \#\langle s_{n-1}\rangle \leqslant \#\langle s'_n\rangle.$$

Remark 1. Observe that from $|s_n - s'_n| = k3^{-n}$, it follows that if $s_{n+1} \in \text{supp}\ \mu_{n+1}$ and $s_{n+1} = s_n + \frac{1}{3^{n+1}}$ then s_{n+1} can not be represented in the forms

$$s_{n+1} = s'_n + \frac{0}{3^{n+1}}, \text{ or } s_{n+1} = s''_n + \frac{6}{3^{n+1}},$$

where $s_n, s'_n, s''_n \in \text{supp}\ \mu_n$. Thus, for any $s_{n+1} \in \text{supp}\ \mu_{n+1}$ has at most two representations throught points in supp μ_n.

2.2. Claim. Assume that $s_n, s'_n \in \text{supp}\ \mu_n, n \geq 3$. Then we have

a) If $s_n - s'_n = \frac{1}{3^n}$, then there are three following cases for the representation of s_n, s'_n:

1. $s_n = s_{n-1} + \frac{1}{3^n}$; $s'_n = s_{n-1} + \frac{0}{3^n}$,
2. $s_n = s_{n-2} + \frac{6}{3^{n-1}} + \frac{1}{3^n}$; $s'_n = s'_{n-2} + \frac{1}{3^{n-1}} + \frac{6}{3^n}$, or
3. $s_n = s_{n-2} + \frac{0}{3^{n-1}} + \frac{1}{3^n}$; $s'_n = s'_{n-2} + \frac{1}{3^{n-1}} + \frac{6}{3^n}$,

where $s_{n-1} \in \text{supp}\ \mu_{n-1}$ and $s_{n-2}, s'_{n-2} \in \text{supp}\ \mu_{n-2}$.

b) If $s_n - s'_n = \frac{2}{3^n}$ then there are four following cases for the representation of s_n, s'_n:

1. $s_n = s_{n-2} + \frac{0}{3^{n-1}} + \frac{6}{3^n}$; $s'_n = s'_{n-2} + \frac{1}{3^{n-1}} + \frac{1}{3^n}$,
2. $s_n = s_{n-2} + \frac{1}{3^{n-1}} + \frac{0}{3^n}$; $s'_n = s'_{n-2} + \frac{0}{3^{n-1}} + \frac{1}{3^n}$,
3. $s_n = s_{n-2} + \frac{6}{3^{n-1}} + \frac{6}{3^n}$; $s'_n = s'_{n-2} + \frac{1}{3^{n-1}} + \frac{1}{3^n}$, or
4. $s_n = s_{n-2} + \frac{1}{3^{n-1}} + \frac{0}{3^n}$; $s'_n = s'_{n-2} + \frac{6}{3^{n-1}} + \frac{1}{3^n}$,

where $s_{n-1} \in \text{supp } \mu_{n-1}$ and $s_{n-2}, s'_{n-2} \in \text{supp } \mu_{n-2}$.
Proof. Let $s_n = \sum_{i=1}^n 3^{-i} x_i$ and $s'_n = \sum_{i=1}^n 3^{-i} x'_i$, $x_i, x'_i \in D$.

a) If $s_n - s'_n = \frac{1}{3^n}$ then $3^{n-1}(x_1 - x'_1) + 3^{n-2}(x_2 - x'_2) + \ldots + 3(x_{n-1} - x'_{n-1}) + x_n - x'_n = 1$, which implies $s_n - s'_n \equiv 1 \pmod 3$, hence $x_n - x'_n = 1$ or $x_n - x'_n = -5$.

For $x_n - x'_n = 1$ we have $x_n = 1, x'_n = 0$. This is the case 1.a.

For $x_n - x'_n = -5$ we have $x_n = 1, x'_n = 6$ and

$$3^{n-2}(x_1 - x'_1) + \ldots + 3(x_{n-2} - x'_{n-2}) + x_{n-1} - x'_{n-1} = 2,$$

which implies $s_{n-1} - s'_{n-1} \equiv 2 \pmod 3$, hence $x_{n-1} - x'_{n-1} = 5$ $(x_{n-1} = 6, x'_{n-1} = 1)$ or $x_{n-1} - x'_{n-1} = -1$ $(x_{n-1} = 0, x'_{n-1} = 1)$ are the cases 2.a, 3.a respectively.

b) The proof is similar to a).

Consequence 2. *Let $s_n < s'_n < s''_n$ be three arbitrary consecutive points in suppμ_n. Then either $s'_n - s_n$ or $s''_n - s'_n$ is not $\frac{1}{3^n}$.*

The following fact provides a useful formula for calculating the local dimention.

2.3. Proposition. *For $s \in$ supp μ, we have*

$$\alpha(s) = \lim_{n \to \infty} \frac{|\log \mu_n(s_n)|}{n \log 3},$$

provided that the limit exists. Otherwise, by taking the upper and lower limits respectively we get the formulas for $\overline{\alpha}(s)$ and $\underline{\alpha}(s)$.

We first prove:

2.4. Lemma. *For any two consecutive points s_n and s'_n in supp μ_n we have*

$$\frac{\mu_n(s_n)}{\mu_n(s'_n)} \leqslant n.$$

Proof. By (2) it is sufficient to show that $\frac{\#\langle s_n \rangle}{\#\langle s'_n \rangle} \leqslant n$. We will prove the inequality by induction. Clearly the inequality holds for $n = 1$. Suppose that it is true for all $n \leqslant k$. Let $s_{k+1} > s'_{k+1}$ be two arbitrary consecutive points in supp μ_{n+1}. Write

$$s_{k+1} = s_k + \frac{x_{k+1}}{3^{k+1}}, \; s_k \in \text{supp } \mu_k, x_{k+1} \in D.$$

We consider the following cases for x_{k+1}:
Case 1. $x_{k+1} = 6$. $s_{k+1} = s_k + \frac{6}{3^{k+1}} = s_k + \frac{2}{3^k}$. Let $s'_k \in \text{supp } \mu_k$ be the smallest value larger than s_k.

1.a) If $s'_k = s_k + \frac{1}{3^k}$ then $s'_{k+1} = s'_k + \frac{1}{3^{k+1}}$, hence by Consequence 1.a, we have $\#\langle s'_{k+1} \rangle = \#\langle s'_k \rangle$. Note that if s_{k+1} has a other representation, $s'_{k+1} = s''_k + \frac{0}{3^{k+1}}, s''_k \in$ supp μ_k, then s_k and s''_k are two consecutive points in supp μ_k and $s_k < s'_k < s''_k$, a contradiction. It follows that $\#\langle s_{k+1} \rangle = \#\langle s_k \rangle$. Therefore

$$\frac{\#\langle s_{k+1} \rangle}{\#\langle s'_{k+1} \rangle} = \frac{\#\langle s_k \rangle}{\#\langle s'_k \rangle} \leqslant k < k + 1.$$

1.b) If $s'_k \geq s_k + \frac{2}{3^k} = s_{k+1}$. So s_{k+1} has at most two representations through s_k and s'_k($s_{k+1} = s_k + \frac{6}{3^{k+1}}$ and $s_{k+1} = s'_k + \frac{0}{3^{k+1}}$). It follows that

$$\#\langle s_{k+1}\rangle \leqslant \#\langle s_k\rangle + \#\langle s'_k\rangle.$$

Since $s_k < s_k + \frac{1}{3^{k+1}} < s_{k+1} \leqslant s'_k$, $s'_{k+1} \in (s_k, s_{k+1})$. On the other hand s_k, s'_k are two consecutive points in supp μ_k, so $s'_{k+1} \notin$ supp μ_k. It follows that
If $s''_k + \frac{6}{3^{k+1}} < s_k + \frac{1}{3^{k+1}}$ for $s''_k \in$ supp μ_k with $s''_k < s_k$ then $s'_{k+1} = s_k + \frac{1}{3^{k+1}}$. Therefore

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} \leqslant \frac{\#\langle s_k\rangle + \#\langle s'_k\rangle}{\#\langle s_k\rangle} \leqslant k+1.$$

If there exists $s''_k \in$ supp μ_k such that $s_k + \frac{1}{3^{k+1}} < s''_k + \frac{6}{3^{k+1}} < s_{k+1}(s''_k < s_k)$ then $s'_{k+1} = s''_k + \frac{6}{3^{k+1}}$ and $0 < s_k - s''_k < \frac{5}{3^{k+1}} < \frac{2}{3^k}$, so $s_k = s''_k + \frac{1}{3^k}$. By Consequece1.b), $\#\langle s'_{k+1}\rangle = \#\langle s''_k\rangle \geq \#\langle s_k\rangle$. Therefore

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} \leqslant \frac{\#\langle s_k\rangle + \#\langle s'_k\rangle}{\#\langle s_k\rangle} \leqslant k+1.$$

Case 2. $x_{k+1} = 1$. $s_{k+1} = s_k + \frac{1}{3^{k+1}}$. Then $s'_{k+1} = s_k + \frac{0}{3^{k+1}}$. If there exists $s'_k \in$ supp μ_k such that $s'_{k+1} = s'_k + \frac{6}{3^{k+1}}$ then s'_k, s_k are two consecutive points in supp μ_k(because $s_k - s'_k = 2/3^k$). Therefore

$$\frac{\#\langle s'_{k+1}\rangle}{\#\langle s_{k+1}\rangle} = \frac{\#\langle s'_{k+1}\rangle}{\#\langle s_k\rangle} \leqslant \frac{\#\langle s_k\rangle + \#\langle s'_k\rangle}{\#\langle s_k\rangle} \leqslant k+1.$$

Case 3. $x_{k+1} = 0$. $s_{k+1} = s_k + \frac{0}{3^{k+1}}$. Note that if s_{k+1} has other representation, $s_{k+1} = s'_k + \frac{6}{3^{k+1}}$ then it was considered in the Case 1. So we may suppose that

$$s_{k+1} \neq s_k + \frac{6}{3^{k+1}} \text{ for all } s_k \in \text{supp } \mu_k. \tag{3}$$

Then we have $\#\langle s_{k+1}\rangle = \#\langle s_k\rangle$. Write

$$s'_{k+1} = s'_k + \frac{x'_{k+1}}{3^{k+1}},\ x'_{k+1} \in D.$$

Since $s_{k+1} = s_k + \frac{0}{3^{k+1}} \in$ supp μ_k and $x'_{k+1} \neq 0$, $x'_{k+1} = 1$ or $x'_{k+1} = 6$. Which implies $\#\langle s'_{k+1}\rangle = \#\langle s'_k\rangle$. We claim that s_k and s'_k are two consecutive points in supp μ_k.

In fact, if there exists $s''_k \in$ supp μ_k such that $s'_k < s''_k < s_k = s_{k+1}$, then

$$s'_{k+1} = s'_k + \frac{6}{3^{k+1}} \tag{4}$$

(If it is not the case, $s'_{k+1} = s'_k + \frac{1}{3^{k+1}} < s''_k < s_k = s_{k+1}$, then s'_{k+1} and s_{k+1} are not consecutive).

Since s'_{k+1} and s_{k+1} are two consecutive points, $s''_k < s'_{k+1} = s'_k + \frac{6}{3^{k+1}} = s'_k + \frac{2}{3^k}$, hence

$$s''_k = s'_k + \frac{1}{3^k}. \tag{5}$$

From Consequence 2 and (3),

$$s''_k + \frac{6}{3^{k+1}} = s''_k + \frac{2}{3^k} < s_k = s_{k+1}. \tag{6}$$

From (4), (5) and (6) we get $s'_{k+1} = s'_k + \frac{6}{3^{k+1}} = s'_k + \frac{2}{3^k} = s''_k + \frac{1}{3^k} < s''_k + \frac{6}{3^{k+1}} < s_k = s_{k+1}$, a contradiction to s'_{k+1} and s_{k+1} are two consecutive points.

Therefore

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} = \frac{\#\langle s_k\rangle}{\#\langle s'_k\rangle} \leqslant k < k+1.$$

Proof of Proposition 2.3. We first show that for $r$given ≥ 1 and for any $s \in \operatorname{supp} \mu$ if there exists $\lim\limits_{n\to\infty} \frac{\log \mu(B_{r3^{-n}}(s))}{\log(r3^{-n})}$, then

$$\alpha(s) = \lim_{n\to\infty} \frac{\log \mu(B_{r3^{-n}}(s))}{\log(r3^{-n})} = \lim_{n\to\infty} \frac{\log \mu(B_{r3^{-n}}(s))}{\log 3^{-n}}. \tag{7}$$

Indeed, for $0 < h \leqslant 1$ take n such that $3^{-n-1} < \frac{h}{r} \leqslant 3^{-n}$. Then

$$\frac{\log \mu(B_{r3^{-n}}(s))}{\log(r3^{-n-1})} \leqslant \frac{\log \mu(B_h(s))}{\log h} \leqslant \frac{\log \mu(B_{r3^{-n-1}}(s))}{\log(r3^{-n})}.$$

Since $\lim\limits_{n\to\infty} \frac{\log(r3^{-n-1})}{\log(r3^{-n})} = 1$, we have

$$\lim_{n\to\infty} \frac{\log \mu(B_{r3^{-n}}(s))}{\log(r3^{-n-1})} = \lim_{n\to\infty} \frac{\log \mu(B_{r3^{-n-1}}(s))}{\log(r3^{-n})} = \lim_{n\to\infty} \frac{\log \mu(B_{r3^{-n}}(s))}{\log(r3^{-n})}.$$

Therefore, (7) follows. Since

$$|S - S_n| \leqslant 6\sum_{i=1}^{\infty} 3^{-n-i} = 3.3^{-n},$$

we have

$$\begin{aligned}
\mu(B_{3^{-n}}(s)) &= \operatorname{Prob}(|S - s| \leqslant 3^{-n}) \\
&\leqslant \operatorname{Prob}(|S_n - s| \leqslant 3^{-n} + 3.3^{-n} = 4.3^{-n}) \\
&= \mu_n(B_{r3^{-n}}(s)),
\end{aligned} \tag{8}$$

where $r = 4$.

Similarly, we obtain

$$\mu_n(B_{r3^{-n}}(s)) \leqslant \mu(B_{(r+3)3^{-n}}(s)).$$

From the latter and (8) we get

$$\frac{\log \mu(B_{(r+3)3^{-n}}(s))}{\log 3^{-n}} \leqslant \frac{\log \mu_n(B_{r3^{-n}}(s))}{\log 3^{-n}} \leqslant \frac{\log \mu(B_{3^{-n}}(s))}{\log 3^{-n}}.$$

Letting $n \to \infty$, by (7) we obtain

$$\alpha(s) = \lim_{n\to\infty} \frac{\log \mu_n(B_{r3^{-n}}(s))}{\log 3^{-n}}. \tag{9}$$

Observe that $B_{r3^{-n}}(s)$ contains s_n and at most six consecutive points in supp μ_n (because $2r = 8$ and by Consequence 2). By Lemma 2.4,

$$\frac{\log \mu_n(s_n)}{\log 3^{-n}} \geq \frac{\log \mu_n(B_{r3^{-n}}(s))}{\log 3^{-n}} \geq \frac{\log[6n^5\mu_n(s_n)]}{\log 3^{-n}}.$$

From the latter and (9) we get

$$\alpha(s) = \lim_{n\to\infty} \frac{\log \mu_n(s_n)}{\log 3^{-n}} = \lim_{n\to\infty} \frac{|\log \mu_n(s_n)|}{n \log 3}.$$

The proposition is proved.

For each infinite sequence $x = (x_1, x_2, \dots) \in D^\infty$ defines a point $s \in \operatorname{supp} \mu$ by

$$s = S(x) := \sum_{n=1}^{\infty} 3^{-n} x_n.$$

Let

$$x = (x_1, x_2, \dots) = (0, 6, 0, 6, \dots), \quad \text{i.e., } x_{2k-1} = 0, x_{2k} = 6, k = 1, 2, \dots \tag{10}$$

Then we have

2.5. Claim. For $x = (x_1, x_2, \dots) \in D^\infty$ is defined by (10), we have

a)

$$\#\langle s_{2n}\rangle = \#\langle s_{2n-1}\rangle;$$

b)

$$\#\langle s_{2(n+1)}\rangle = \#\langle s_{2n}\rangle + \#\langle s_{2(n-1)}\rangle, \tag{11}$$

for every $n \geq 2$, where s_n denotes n- partial sum of $s = S(x)$.

Proof. a) Observe that $\#\langle s_{2n}\rangle \geq \#\langle s_{2n-1}\rangle$. On the other hand, let $(x'_1, x'_2, \dots, x'_{2n}) \in \langle s_{2n}\rangle$. If $x'_{2n} \neq 6$, then by Claim 2.1, $x'_{2n} = 0$. It follows that $s'_{2n-1} - s_{2n-1} = \frac{2}{3^{2n-1}}$, where $s'_{2n-1} = \sum_{i=1}^{2n-1} 3^{-i} x'_i$. From Claim 2.2.b), it follows that $x_{2n-1} = 1$, a contradiction to $x_{2n-1} = 0$. Thus $x'_{2n} = 6$, which implies $(x'_1, x'_2, \dots, x'_{2n-1}) \in \langle s_{2n-1}\rangle$. That means

$$\#\langle s_{2n-1}\rangle \geq \#\langle s_{2n}\rangle.$$

Therefore

$$\#\langle s_{2n}\rangle = \#\langle s_{2n-1}\rangle.$$

b) For any element $(x'_1, x'_2, \dots, x'_{2n}, x'_{2n+1}, x'_{2n+2}) \in \langle s_{2n+2}\rangle$, from the proof of a) we have $(x'_1, x'_2, \dots, x'_{2n+1}) \in \langle s_{2n+1}\rangle$. So, by Claim 2.1, $x'_{2n+1} = 0$ or $x'_{2n+1} = 6$ (because $x_{2n+1} = 0$).

If $x'_{2n+1} = 0$ then $(x'_1, x'_2, \dots, x'_{2n}) \in \langle s_{2n}\rangle$.

If $x'_{2n+1} = 6$, since $(x'_1, x'_2, \dots, x'_{2n}, 6, 6) \approx (x_1, x_2, \dots, x_{2n-1}, x_{2n}, 0, 6)$, $s_{2n} - s'_{2n} = \frac{2}{3^{2n}}$, where $s'_{2n} = \sum_{i=1}^{2n} 3^{-i}x'_i$. By Claim 2.2.b) and $x_{2n} = 6$ we have $x'_{2n-1} = x'_{2n} = 1$, which implies $(x'_1, x'_2, \dots, x'_{2n-2}) \in \langle s_{2n-2}\rangle$ (because $(0, 6, 0) \approx (1, 1, 6)$). Let

$$A = \{(x'_1, x'_2, \dots, x'_{2n-2}, x'_{2n-1}, x'_{2n}, 0, 6) : (x'_1, x'_2, \dots, x'_{2n}) \in \langle s_{2n}\rangle\},$$

$$B = \{(y'_1, y'_2, \dots, y'_{2n-2}, 1, 1, 6, 6) : (y'_1, y'_2, \dots, y'_{2n-2}) \in \langle s_{2n-2}\rangle\}.$$

From the above arguments we have

$$A \cup B = \langle s_{2n+2}\rangle \quad \text{and} \quad A \cap B = \emptyset.$$

Therefore

$$\#\langle s_{2(n+1)}\rangle = \#A + \#B = \#\langle s_{2n}\rangle + \#\langle s_{2(n-1)}\rangle.$$

The lemma is proved.

Consequence 3. *For $s \in supp\ \mu$ is defined as in Claim 2.5 we have*

$$\#\langle s_{2n}\rangle = \#\langle s_{2n-1}\rangle = \frac{\sqrt{5}}{5}[(\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1}], \tag{12}$$

for every $n \geq 1$.

Proof. It is easy to see that (12) satisfies (11).

2.6. Claim. For $s \in \operatorname{supp} \mu$ is defined as in Claim 2.5 we have

$$\alpha(s) = 1 - \frac{\log(1+\sqrt{5}) - \log 2}{2\log 3}.$$

Proof. For $n \geq 2$ take $k \in \mathbb{N}$ such that $2k \leqslant n < 2(k+1)$. By (12),

$$\frac{\sqrt{5}}{5}(a_1^{k+1} - a_2^{k+1}) \leqslant \#\langle s_n\rangle \leqslant \frac{\sqrt{5}}{5}(a_1^{k+2} - a_2^{k+2}),$$

where $a_1 = \frac{1+\sqrt{5}}{2}, a_2 = \frac{1-\sqrt{5}}{2}$.
It follows that

$$\frac{|\log \frac{\sqrt{5}}{5}(a_1^{k+2} - a_2^{k+2})3^{-2k}|}{2(k+1)\log 3} \leqslant \frac{|\log \mu_n(s_n)|}{n\log 3} \leqslant \frac{|\log \frac{\sqrt{5}}{5}(a_1^{k+1} - a_2^{k+1})3^{-2k-2}|}{2k\log 3}.$$

Since

$$\lim_{k\to\infty} \frac{|\log \frac{\sqrt{5}}{5}(a_1^{k+2} - a_2^{k+2})3^{-2k}|}{2(k+1)\log 3} = \lim_{k\to\infty} \frac{|\log \frac{\sqrt{5}}{5}(a_1^{k+1} - a_2^{k+1})3^{-2k-2}|}{2k\log 3} = 1 - \frac{\log a_1}{2\log 3},$$

by Proposition 2.3 we get

$$\alpha(s) = 1 - \frac{\log(1+\sqrt{5}) - \log 2}{2\log 3}.$$

The claim is proved.

2.7. Claim. Let $x = (x_1, x_2, \dots)$ be a sequence defined by (10). Then we have

$$3\#\langle s_{2n-1}\rangle < 2\#\langle s_{2n+1}\rangle \text{ for every } n,$$

where $s = S(x)$ and s_n denotes n-partial sum of s.

Proof. Observe that the assertion holds for $n = 1, 2$. For $n \geq 3$, by Claim 2.5 we have

$$\begin{aligned} 2\#\langle s_{2n+1}\rangle &= 2\#\langle s_{2n-1}\rangle + 2\#\langle s_{2n-3}\rangle \\ &= 3\#\langle s_{2n-1}\rangle - \#\langle s_{2n-1}\rangle + 2\#\langle s_{2n-3}\rangle \\ &= 3\#\langle s_{2n-1}\rangle - \#\langle s_{2n-3}\rangle - \#\langle s_{2n-5}\rangle + 2\#\langle s_{2n-3}\rangle \\ &= 3\#\langle s_{2n-1}\rangle + \#\langle s_{2n-3}\rangle - \#\langle s_{2n-5}\rangle > 3\#\langle s_{2n-1}\rangle. \end{aligned}$$

The claim is proved.

2.8. Claim. Assume that $s_{n+1} \in \text{supp } \mu_{n+1}$ has two representations through points in supp $\mu_n (n > 3)$. Then, either

$$\#\langle s_{n+1}\rangle = \#\langle s_{n-1}\rangle + \#\langle s_{n-3}\rangle \text{ for some } s_{n-1} \in \text{supp } \mu_{n-1} \text{ and some } s_{n-3} \in \text{supp } \mu_{n-3},$$

or

$$\#\langle s_{n+1}\rangle \leqslant 2\#\langle s_{n-2}\rangle \text{ for some } s_{n-2} \in \text{supp } \mu_{n-2}.$$

Proof. Let $s_{n+1} = s_n + \frac{0}{3^{n+1}} = s'_n + \frac{6}{3^{n+1}}$, which implies $s_n - s'_n = \frac{2}{3^n}$, so by Claim 2.2.b), $x'_n = 1, x_n = 0$ or $x_n = 6$. We consider the case $x_n = 0$. The case $x_n = 6$ is similar. We have

$$s_{n+1} = s_{n-1} + \frac{0}{3^n} + \frac{0}{3^{n+1}} = s'_{n-1} + \frac{1}{3^n} + \frac{6}{3^{n+1}}. \tag{13}$$

We claim that s_n has only one representation through point $s_{n-1} \in \text{supp } \mu_{n-1}$. In fact, if it is not the case, $s_n = s_{n-1} + \frac{0}{3^n} = s''_{n-1} + \frac{6}{3^n}$, then

$$s_{n+1} = s_{n-1} + \frac{0}{3^n} + \frac{0}{3^{n+1}} = s''_{n-1} + \frac{6}{3^n} + \frac{0}{3^{n+1}} = s'_{n-1} + \frac{1}{3^n} + \frac{6}{3^{n+1}},$$

which implies $s_{n-1} - s'_{n-1} = s'_{n-1} - s''_{n-1} = \frac{1}{3^{n-1}}$. a contradiction to Consequence 2. Hence,

$$\#\langle s_{n+1}\rangle = \#\langle s_{n-1}\rangle + \#\langle s'_{n-1}\rangle.$$

From (13) yield $s_{n-1} - s'_{n-1} = \frac{1}{3^{n-1}}$, by Claim 2.2.a), $x_{n-1} = 1$. So that , by Consequence 1.a), $\#\langle s_{n-1}\rangle = \#\langle s_{n-2}\rangle$. Therefore

$$\#\langle s_{n+1}\rangle = \#\langle s_{n-2}\rangle + \#\langle s'_{n-1}\rangle.$$

Consider the following cases.

1. If s'_{n-1} has only one representation through some point $s'_{n-2} \in \operatorname{supp} \mu_{n-2}$ then $\#\langle s'_{n-1}\rangle = \#\langle s'_{n-2}\rangle$. Without loss of generality we may assume that $\#\langle s_{n-2}\rangle \geq \#\langle s'_{n-2}\rangle$. Then

$$\#\langle s_{n+1}\rangle = \#\langle s_{n-2}\rangle + \#\langle s'_{n-2}\rangle \leqslant 2\#\langle s_{n-2}\rangle.$$

2. If s'_{n-1} has two representations through points in supp μ_{n-2}, $s'_{n-1} = s_{n-2} + \frac{0}{3^{n-1}} = s'_{n-2} + \frac{6}{3^{n-1}}$, then

$$\begin{aligned} s_{n+1} = s_{n-2} + \frac{1}{3^{n-1}} + \frac{0}{3^n} + \frac{0}{3^{n+1}} &= s_{n-2} + \frac{0}{3^{n-1}} + \frac{1}{3^n} + \frac{6}{3^{n+1}} \\ &= s'_{n-2} + \frac{6}{3^{n-1}} + \frac{1}{3^n} + \frac{6}{3^{n+1}}. \end{aligned}$$

Since $(1,0,0) \approx (0,1,6)$, $s_{n-2} = s''_{n-2}$, and so $s_{n-2} - s'_{n-2} = \frac{2}{3^{n-2}}$. Hence, by Claim 2.2.b), $x'_{n-2} = 1$. Thus, $s'_{n-2} = s'_{n-3} + \frac{1}{3^{n-2}}$.
We check that s_{n-2} has only one representation through some point $s_{n-3} \in \operatorname{supp} \mu_{n-3}$. If it is not the cases $s_{n-2} = s_{n-3} + \frac{0}{3^{n-2}} = s''_{n-3} + \frac{6}{3^{n-2}}$, then

$$\begin{aligned} s_{n+1} &= s_{n-3} + \frac{0}{3^{n-2}} + \frac{1}{3^{n-1}} + \frac{0}{3^n} + \frac{0}{3^{n+1}} \\ &= s''_{n-3} + \frac{6}{3^{n-2}} + \frac{1}{3^{n-1}} + \frac{0}{3^n} + \frac{0}{3^{n+1}} \\ &= s'_{n-3} + \frac{1}{3^{n-2}} + \frac{6}{3^{n-1}} + \frac{1}{3^n} + \frac{6}{3^{n+1}}, \end{aligned}$$

which implies $s_{n-3} - s'_{n-3} = s'_{n-3} - s''_{n-3} = \frac{1}{3^{n-3}}$. Which is a contradiction to Consequence 2. So, $\#\langle s_{n-2}\rangle = \#\langle s_{n-3}\rangle$. Therefore

$$\#\langle s_{n+1}\rangle = \#\langle s_{n-3}\rangle + \#\langle s'_{n-1}\rangle.$$

The claim is proved.

2.9. Claim. Let $k \geq 3$ be a natural number such that

$$\#\langle t_{2n+1}\rangle \leqslant \#\langle s_{2n+1}\rangle \text{ for all } n \leqslant k \text{ and for every } t_{2n+1} \in \operatorname{supp} \mu_{2n+1}.$$

Then

$$2\#\langle t_{2n}\rangle \leqslant \#\langle s_{2n+1}\rangle + \#\langle s_{2n-1}\rangle \text{ for all } n \leqslant k \text{ and for every } t_{2n} \in \operatorname{supp} \mu_{2n},$$

where s is defined as in Claim 2.5 and s_n denotes n-partial sum of s.
Proof. Observe that, if t_{2n} has only one representation through point $t_{2n-1} \in \operatorname{supp} \mu_{2n-1}$ then the claim is true. Suppose that t_{2n} has two representations through points in supp μ_{2n-1}, by Claim 2.8, either $\#\langle t_{2n}\rangle = \#\langle t_{2n-2}\rangle + \#\langle t_{2n-4}\rangle$ or $\#\langle t_{2n}\rangle \leqslant 2\#\langle t_{2n-3}\rangle$.
1. Let $\#\langle t_{2n}\rangle = \#\langle t_{2n-2}\rangle + \#\langle t_{2n-4}\rangle$. Putting

$$t_{2n+1} = t_{2n-2} + \frac{0}{3^{2n-1}} + \frac{6}{3^{2n}} + \frac{0}{3^{2n+1}}\ ,\ t_{2n-1} = t_{2n-4} + \frac{0}{3^{2n-3}} + \frac{6}{3^{2n-2}} + \frac{0}{3^{2n-1}}$$

we have

$$\#\langle t_{2n+1}\rangle \geq 2\#\langle t_{2n-2}\rangle \ , \ \#\langle t_{2n-1}\rangle \geq 2\#\langle t_{2n-4}\rangle.$$

It follows that

$$\begin{aligned} 2\#\langle t_{2n}\rangle &= 2\#\langle t_{2n-2}\rangle + 2\#\langle t_{2n-4}\rangle \\ &\leqslant \#\langle t_{2n+1}\rangle + \#\langle t_{2n-1}\rangle \leqslant \#\langle s_{2n+1}\rangle + \#\langle s_{2n-1}\rangle. \end{aligned}$$

2. $\#\langle t_{2n}\rangle \leqslant 2\#\langle t_{2n-3}\rangle$. By Claims 2.5 and 2.7 we get

$$\begin{aligned} 2\#\langle t_{2n}\rangle \leqslant 4\#\langle t_{2n-3}\rangle &\leqslant 4\#\langle s_{2n-3}\rangle \\ &= \#\langle s_{2n-3}\rangle + 3\#\langle s_{2n-3}\rangle \\ &\leqslant \#\langle s_{2n-3}\rangle + 2\#\langle s_{2n-1}\rangle. \\ &= \#\langle s_{2n+1}\rangle + \#\langle s_{2n-1}\rangle. \end{aligned}$$

The claim is proved.

We say that $x = (x_1, x_2, \dots, x_n) \in D^n$ is a *maximal sequence* if

$$\#\langle t_n\rangle \leqslant \#\langle s_n\rangle \text{ for every } t_n \in \text{supp } \mu_n,$$

where $s_n = \sum_{i=1}^{n} 3^{-i} x_i$.
The following fact given an estimation for the greatest lower bound of local dimension.

2.10. Proposition. *For every* $n, (x_1, x_2, \dots, x_{2n+1}) = (0, 6, 0, 6, \dots, 0, 6, 0)$ *is a maximal sequence.*

Proof. We will prove the proposition by induction. By Claim 2.8, it is straightforward to check that the assertion holds for $n = 1, 2, 3$ ($\#\langle s_3\rangle = 2, \#\langle s_5\rangle = 3, \#\langle s_7\rangle = 5$). Suppose that it is true for all $n \leqslant k (k \geq 3)$. We show that the proposition is true for $n = k + 1$. Let $t_{2(k+1)+1} = t_{2k+3}$ be an arbitrary point in supp μ_{2k+3}. Clearly the assertion holds if t_{2k+3} has only one representation through some point in supp μ_{2k+2}. If t_{2k+3} has two representations through points in supp μ_{2k+2} then by Claim 2.8 we have two following cases:

Case 1. $\#\langle t_{2k+3}\rangle = \#\langle t_{2k+1}\rangle + \#\langle t_{2k-1}\rangle$ for some $t_{2k+1} \in$ supp μ_{2k+1} and some $t_{2k-1} \in$ supp μ_{2k-1}. Then, by Claim 2.5 we get

$$\#\langle t_{2k+3}\rangle \leqslant \#\langle s_{2k+1}\rangle + \#\langle s_{2k-1}\rangle = \#\langle s_{2k+3}\rangle.$$

Case 2. $\#\langle t_{2k+3}\rangle \leqslant 2\#\langle t_{2k}\rangle$ for some $t_{2k} \in$ supp μ_{2k}. Then, by Claims 2.5 and 2.9 we have

$$\#\langle t_{2k+3}\rangle \leqslant 2\#\langle t_{2k}\rangle \leqslant \#\langle s_{2k+1}\rangle + \#\langle s_{2k-1}\rangle = \#\langle s_{2k+3}\rangle.$$

Therefore, $(x_1, x_2, \dots, x_{2n+1}) = (0, 6, 0, 6, \dots, 0, 6, 0)$ is a maximal sequence. The proposition is proved.

3. Proof of The Main Theorem

We call an infinite sequence $x = (x_1, x_2, \dots) \in D^\infty$ a *prime sequence* if $\#\langle s_n \rangle = 1$ for every n, where $s_n = \sum_{i=1}^{n} 3^{-i} x_i$.

3.1. Claim. $\overline{\alpha} = 1$, $\underline{\alpha} = 1 - \frac{\log(1+\sqrt{5})-\log 2}{2\log 3} = 1 - \frac{\log a_1}{2\log 3}$.

Proof. For any prime sequence $x = (x_1, x_2, \dots)$ we have $\#\langle s_n \rangle = 1$ for every n, where $s_n = \sum_{i=1}^{n} 3^{-i} x_i$. Therefore, by Proposition 2.3 we get

$$\overline{\alpha} = \overline{\alpha}(s) = \lim_{n\to\infty} \frac{|\log \mu_n(s_n)|}{n\log 3} = 1,$$

where $s = S(x)$.

From Claim 2.6 we have

$$\underline{\alpha} \leqslant 1 - \frac{\log a_1}{2\log 3}.$$

For any $t \in$ supp μ, by Proposition 2.10 $\#\langle t_{2n+1} \rangle \leqslant \#\langle s_{2n+1} \rangle = \frac{\sqrt{5}}{5}(a_1^{n+2} - a_2^{n+2})$ for evrey n, we have

$$\lim_{n\to\infty} \frac{|\log \mu_{2n+1}(t_{2n+1})|}{(2n+1)\log 3} \geq \lim_{n\to\infty} \frac{|\log \frac{\sqrt{5}}{5}(a_1^{n+2} - a_2^{n+2})3^{-2n-1}|}{2(n+1)\log 3} = 1 - \frac{\log a_1}{2\log 3}, \tag{14}$$

where t_n be n- partial sum of t. On the other hand, since $\#\langle t_{2n} \rangle \leqslant \#\langle s_{2n+1} \rangle$,

$$\lim_{n\to\infty} \frac{|\log \mu_{2n}(t_{2n})|}{(2n)\log 3} \geq \lim_{n\to\infty} \frac{|\log \frac{\sqrt{5}}{5}(a_1^{n+2} - a_2^{n+2})3^{-2n}|}{2(n)\log 3} = 1 - \frac{\log a_1}{2\log 3}. \tag{15}$$

Combinating (14) and (15) we get

$$\underline{\alpha} \geq 1 - \frac{\log a_1}{2\log 3}.$$

Therefore

$$\underline{\alpha} = 1 - \frac{\log a_1}{2\log 3} = 1 - \frac{\log(1+\sqrt{5}) - \log 2}{2\log 3}.$$

The claim is proved.

To complete the proof of our Main Theorem it remains to show that $E = [1 - \frac{\log(1+\sqrt{5})-\log 2}{2\log 3}, 1]$, i.e., for any $\beta \in (1 - \frac{\log a_1}{2\log 3}, 1)$ there exists $s \in$ supp μ for which $\alpha(s) = \beta$. Let $r = 2(1-\beta)\frac{\log 3}{\log a_1}$. It is easy to see that $0 < r < 1$.
For $i = 1, 2, \dots$, define

$$k_i = \begin{cases} 2i+1 & \text{if } i \text{ is odd;} \\ [\frac{2i(1-r)}{r}] & \text{if } i \text{ is even,} \end{cases}$$

where $[x]$ denotes the largest integer $\leqslant x$.
Let $n_j = \sum_{i=1}^{j} k_i$ and let

$$E_j = \{i : i \leqslant j \text{ and i is even}\}\ ;\ O_j = \{i : i \leqslant j \text{ and i is odd}\},$$

$$e_j = \sum_{i\in E_j} k_i \ ; \ o_j = \sum_{i\in O_j} k_i.$$

Then $n_j = o_j + e_j$.

3.2. Claim. With the above notation we have

$$\lim_{j\to\infty} \frac{j}{n_j} = 0 \ ; \ \lim_{j\to\infty} \frac{n_{j-1}}{n_j} = 1 \text{ and } \lim_{j\to\infty} \frac{o_j}{n_j} = r.$$

Proof. The proof of the first limit is trivial. The second limit follows from the first one. To prove the third limit, without loss of generality we may assume that $j = 2k+1$. Then we have $o_j = 2\sum_{i=0}^{k}(2i+1) + k + 1 = (k+1)(2k+3)$. Since

$$\frac{4(1-r)}{r}\sum_{i=1}^{k} i - k \leqslant \sum_{i=1}^{k}[\frac{4i(1-r)}{r}] \leqslant \frac{4(1-r)}{r}\sum_{i=1}^{k} i,$$

$$\frac{2(1-r)}{r}k(k+1) - k \leqslant e_j \leqslant \frac{2(1-r)}{r}k(k+1).$$

Hence

$$\frac{(k+1)(2k+3)}{(k+1)(2k+3) + \frac{2(1-r)}{r}k(k+1)} \leqslant \frac{o_j}{n_j} \leqslant \frac{(k+1)(2k+3)}{(k+1)(2k+3) + \frac{2(1-r)}{r}k(k+1) - k}.$$

Therefore

$$\lim_{j\to\infty} \frac{o_j}{n_j} = r.$$

The claim is proved.

We define $s \in \operatorname{supp} \mu$ by $s = S(x)$, where

$$x = (\underbrace{0,6,0,}_{k_1=3}\underbrace{1,1,\ldots,1,}_{k_2}\underbrace{0,6,0,6,0,}_{k_3=5}\underbrace{1,1,\ldots,1,}_{k_3}\ldots). \tag{16}$$

Note that, for $i \in O_j$, from (12),

$$\#\langle s_{k_i}\rangle = \frac{\sqrt{5}}{5}(a_1^{\frac{k_i+3}{2}} - a_2^{\frac{k_i+3}{2}}) = \begin{cases} > \frac{\sqrt{5}}{5}a_1^{\frac{k_i+3}{2}} \\ < \frac{\sqrt{5}}{5}a_1^{\frac{k_i+3}{2}+1}. \end{cases} \tag{17}$$

For $s \in \operatorname{supp} \mu$ is defined (16) and for $n_{j-1} \leqslant n < n_j$ we have

$$\prod_{i\in O_{j-1}} \#\langle s_{k_i}\rangle \leqslant \#\langle s_n\rangle \leqslant \prod_{i\in O_j} \#\langle s_{k_i}\rangle.$$

Hence, by (17) yield

$$(\frac{\sqrt{5}}{5})^{\frac{j-1}{2}} a_1^{\frac{o_{j-1}}{2}+\frac{3}{2}\frac{j-1}{2}} \leqslant \#\langle s_n\rangle \leqslant (\frac{\sqrt{5}}{5})^{\frac{j+1}{2}} a_1^{\frac{o_j}{2}+\frac{5}{2}\frac{j+1}{2}},$$

which implies

$$\frac{\log[(\frac{\sqrt{5}}{5})^{\frac{j-1}{2}} a_1^{\frac{o_{j-1}}{2}+\frac{3}{2}\frac{j-1}{2}}]}{n_j \log 3} \leqslant \frac{\log \#\langle s_n\rangle}{n \log 3} \leqslant \frac{\log[(\frac{\sqrt{5}}{5})^{\frac{j+1}{2}} a_1^{\frac{o_j}{2}+\frac{5}{2}\frac{j+1}{2}}]}{n_{j-1} \log 3}.$$

From Claim 3.1 we get

$$\lim_{n\to\infty} \frac{\log \#\langle s_n\rangle}{n \log 3} = \frac{r}{2}\frac{\log a_1}{\log 3}.$$

Therefore

$$\begin{aligned} \alpha(s) = \lim_{n\to\infty} \frac{|\log \#\langle s_n\rangle 3^{-n}|}{n \log 3} &= 1 - \lim_{n\to\infty} \frac{\log \#\langle s_n\rangle}{n \log 3} \\ &= 1 - \frac{r}{2}\frac{\log a_1}{\log 3} = \beta. \end{aligned}$$

The Main Theorem is proved.

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