A NEW NUMERICAL INVARIANT OF ARTINIAN MODULES OVER NOETHERIAN LOCAL RINGS

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Abstract. Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring the maximal ideal $\mathfrak{m}$ and A an Artinian R-module with Ndim $A = d$. For each system of parameters $\underline{x} = (x_1, ..., x_d)$ of A, we denote by $e(\underline{x}, A)$ the multipility of A with respect to $\underline{x}$. Let $\underline{n} = (n_1, n_2, ..., n_d)$ be a d-tuple of positive integers. The paper concerns to the function of d-variables

$$I(\underline{x}(\underline{n}); A) := \ell_R(0 :_A (x_1^{n_1}, ..., x_d^{n_d})R) - e(x_1^{n_1}, ..., x_d^{n_d}; A),$$

where $\ell_R(-)$ is the length of function. We show in this paper that this function may be not a polynomial in the general case, but the least degree of all upper-bound polynomials for the function is a numerical invariant of A. A characterization for co Cohen-Macaulay modules in term of this new invariant is also given.

Keywords: Artinian module, multiplicity

1. Introduction

Throughout let $(R, \mathfrak{m})$ denote a commutative Noetherian local ring with the maximal ideal $\mathfrak{m}$ and A an Artinian R-module with Ndim $A = d > 0$. For each system of parameters $\underline{x} = (x_1, ..., x_d)$ for A, we denote by $e(\underline{x}; A)$ the multiplicity of A with respect to $\underline{x}$ in the sense of [3]. It has been shown by Kirby in [8] that there exist $q(n) \in \mathbb{Q}[x]$ and $n_0 \in \mathbb{N}$ such that $\ell_R(0 :_R (x_1, ..., x_d)^n A) = q(n)$, $\forall n \geqslant n_0$. It is very important that the degree of $q(n)$ equals d and if a_d is the lead coefficient of $q(n)$ then $a_d \cdot d!$ agrees with $e(\underline{x}; A)$.

Let $\underline{n} = (n_1, ..., n_d) \in \mathbb{N}^d$ and consider

$$I(\underline{x}(\underline{n}); A) := \ell_R(0 :_A (x_1^{n_1}, ..., x_d^{n_d}) - n_1 \cdots n_d \cdot e(\underline{x}; A)$$

as a function on $n_1, ..., n_d$. As shown in Example 3.7, this function, may be not a polynomial on $n_1, ..., n_d$ (even when $\underline{n}$ large enough). The aim of this paper is to show that the above function is still interesting to investigate. First, the least degree of all polynomials bounding this function from above is a numerical invariant of A. Moreover, this invariant carries informations on structure of A. The existence of our invariant is proved in the third section. But before doing this, in the second section, we recall basic terminologies and resuls which are needed later. Some relations between the new invariant with local homology modules are presented in the last section.

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2. Preliminaries

In this section, K is a nonzero Artinian R-module.

2.1. The residuum, residual length and width of Artinian modules

We devote this subsection to recall some basic terminologies and results from [11] and [13].

Let $K = \sum_{i=1}^{h} C_i$ be a minimal secondary representation of K. Set

$\mathfrak{p}_i = \sqrt{0 :_R C_i} (\forall i = 1, ..., h)$, $\operatorname{Att}(K) = \{\mathfrak{p}_1, ..., \mathfrak{p}_h\}$, $K_0 = \sum_{\mathfrak{p}_i \in \operatorname{Att}(K) - \{\mathfrak{m}\}} C_i.$

Then $\operatorname{Att}(K)$ and K_0 are independent of the choice of minimal secondary representation for K. Note that K/K_0 has finite length. This length is called the *residual length* of K and denoted by $R\ell(K)$.

An element $a \in R$ is called *K-coregular element* if $K = aK$. The sequence of elements $a_1, ..., a_n$ of R is called a *K-cosequence* if $0 :_K (a_1, ..., a_n)R \neq 0$ and a_i is $0 :_K (a_1, ..., a_{i-1})R$-coregular element for every $i = 1, ..., n$. We denote by $\operatorname{Width}(K)$ the supremum of lengths of all K-cosequences in $\mathfrak{m}$. It should be mentioned that $a \in R$ is K-coregular if and only if $a \notin \bigcup_{\mathfrak{p} \in \operatorname{Att}(K)} \mathfrak{p}$.

An element $a \in \mathfrak{m}$ is called *pseudo-K-coregular* if $a \notin \bigcup_{\mathfrak{p} \in \operatorname{Att}(K) - \{\mathfrak{m}\}} \mathfrak{p}$. We define *the stability index* $s = s(K)$ of K to be the least integer $i \geqslant 0$ such that $\mathfrak{m}^i K = \mathfrak{m}^{i+1} K$. Note that $\mathfrak{m}^s K = K_0$, and that $a^s K = K_0$ for each pseudo-K-coregular element $a \in \mathfrak{m}$.

2.2. The theory of Noetherian dimension, multiplicity for Artinian modules

We continue in this subsection by reviewing basic definitions and properties on Noetherian dimension and multiplicity of Artinian modules. The interested reader should consult to [9] and [3] for more details.

The *Noetherian dimension* of K, denoted by $\mathrm{N} - \dim_R K$, is defined inductively as follows: when $K = 0$, put $\mathrm{N} - \dim_R K = -1$. Then by induction, for an integer $t \geqslant 0$, we put $\mathrm{N} - \dim_R K = t$ if $\mathrm{N} - \dim_R K < t$ is false and for every ascending sequence $K_0 \subseteq K_1 \subseteq \ldots$ of submodules of K, there exists n_0 such that $\mathrm{N} - \dim_R(K_{n+1}/K_n) < t$ for all $n > n_0$.

A system $\underline{x} = (x_1, \ldots , x_t)$ of elements in $\mathfrak{m}$ is called a *multiplicity system* of K if $\ell_R(0 :_K (x_1, \ldots , x_t)R) < \infty$. Assume that $\mathrm{N} - \dim_R K = d$, then a multiplicity system of K is called a *system of parameter* (s.o.p for short) for K if $t = d$.

Let $\underline{x} = (x_1, \ldots , x_d)$ is a multiplicity system of K. The multiplicity $e(\underline{x}; K)$ of K with respect to $\underline{x}$ is defined inductively as follows: when $d = 0$, we put $e(\emptyset; K) = \ell_R(K)$. Let $d > 0$, then we put

$$e(\underline{x}; K) = e(x_2, \ldots , x_d; 0 :_K x_1) - e(x_2, \ldots , x_d; K/x_1K).$$

3. Main results

The following proposition gives an upper bound polynomial for the function $I(\underline{x}(\underline{n});A)$.

3.1. Proposition. *Let $\underline{x}$ be an s.o.p of A and $\underline{n}=(n_1,...,n_d)\in\mathbb{N}^d$. Then*

$$I(\underline{x}(\underline{n});A)\leqslant n_1\cdots n_d I(x_1,...,x_d;A).$$

Proof: By [6], Lemma 2

$$\ell_R(0:_A y^m)\leqslant m\ell_R(0:_A y), \forall y\in A, \forall m\in\mathbb{N}.$$

Using an induction on d, we get

$$\ell_R(0:_A (x_1^{n_1},...,x_d^{n_d})R\leqslant n_1\cdots n_d\ell_R(0:_A (x_1,...,x_d)R)). \tag{1}$$

On the other hand, according to [3] (3.8),

$$e(\underline{x}(\underline{n});A)=n_1\cdots n_d e(x_1,...,x_d;A). \tag{2}$$

The proposition then comes from (1) and (2).

The proposition 3.1 leads to an immediate consequence as follows.

3.2. Corollary. *If $I(\underline{x}(\underline{n});L)$ is a polynomial, then it is linear in each n_i, $i=1,...,d$.*

The main result of this section is the following.

3.3. Theorem. *Let $\underline{x}=(x_1,...,x_d)$ be a s.o.p of A. Then, the least degree of all polynomials in $n_1,...,n_d$ bounding the function $I(x_1^{n_1},...,x_d^{n_d};A)$ from above does not depend on $\underline{x}$.*

Proof: Denote by $\widehat{R}$ the $\mathfrak{m}$-completion of R. Because A is Artinian R-module, it can be considered as an $\widehat{R}$-module. Note that for each element $a\in R$ and each element $x\in A$, we can see that ax and $\widehat{a}x$ are the same, where $\widehat{a}$ is the image of a by the canonical homomorphism $R\longrightarrow\widehat{R}$. On the other hand, when we regard this $\widehat{R}$-module as R-module by means of the natural map $R\longrightarrow\widehat{R}$, then we recover the orginal R-module structure on A. Furthermore a subset of A is an R-submodule if and only if it is an $\widehat{R}$-submodule (see [2] (10.2.9)). It is easy to see that, for each s.o.p $(x_1,...,x_d)$ of R-module A, $(\widehat{x_1},...,\widehat{x_d})$ forms a s.o.p of $\widehat{R}$-module A. Furthermore,

$$(0:_A(\widehat{x_1},...,\widehat{x_d})\widehat{R})=(0:_A(x_1,...,x_d)R)$$

and therefore,

$$I(x_1^{n_1},...,x_d^{n_d};A)=I(\widehat{x_1},...,\widehat{x_d};A).$$

Hence, it suffices to prove our theorem with assumption that R is complete. In oder to prove this theorem, we need three lemmas in which we always assume that R is complete.

3.4. Lemma. *Let $\underline{x}$ be a s.o.p of A. Then, there exists $k \in \mathbb{N}$ such that*

$$\mathfrak{m}^k \subseteq \underline{x}A + \mathrm{Ann}_R A.$$

Proof: Taking $l \in \mathbb{N}$ such that

$$\mathfrak{m}^l R \subseteq \mathrm{Ann}_R(0 :_A (x_1, ..., x_d)R).$$

Denote by $-^\vee := \mathrm{Hom}_R(-, E(R/\mathfrak{m}))$ the Matlis dual functor, where $E(R/\mathfrak{m})$ is injective hull of $R/\mathfrak{m}$. Then, $A^\vee$ is a Noetherian over R and we have

$$\begin{aligned}
\mathfrak{m}^l R &\subseteq \mathrm{Ann}_R(0 :_A (x_1, ..., x_d)R)^\vee = \mathrm{Ann}_R(A^\vee/(x_1, ..., x_d)A^\vee) \\
&\subseteq \sqrt{\mathrm{Ann}_R(A^\vee/(x_1, ..., x_d)A^\vee)} = \sqrt{(x_1, ..., x_d)R + \mathrm{Ann}_R(A^\vee)} \\
&= \sqrt{(x_1, ..., x_d)R + \mathrm{Ann}_R(A)}.
\end{aligned}$$

Since R is Noetherian, there exists $t \in \mathbb{N}$ such that

$$\left(\sqrt{((x_1, ..., x_d)R + \mathrm{Ann}_R(A))}\right)^t \subseteq ((x_1, ..., x_d)R + \mathrm{Ann}_R(A)).$$

To finish our claim one just set $k = tl$. ■

3.5. Lemma. *Let $x_1, x_2, ..., x_d$ and $y_1, y_2, ..., y_d$ be two s.o.p's of R with $x_1 = y_1, ..., x_{d-1} = y_{d-1}$. Let $n_2, ..., n_d \in \mathbb{N}$. Then there exists a pseudo-A-coregular element, say z_1, such that, for all $n_1 \in \mathbb{N}$,*

$$(x_1^{n_1}, x_2^{n_2}, ..., x_{d-1}^{n_{d-1}}, x_d^{n_d})R = (z_1^{n_1}, x_2^{n_2}, ..., x_d^{n_d})R \tag{3}$$

and

$$(y_1^{n_1}, y_2^{n_2}, ..., y_{d-1}^{n_{d-1}}, y_d^{n_d})R = (z_1^{n_1}, y_2^{n_2}, ..., y_{d-1}^{n_{d-1}}, y_d^{n_d})R. \tag{4}$$

Proof: By Lemma 3.4 we can write

$$\mathfrak{m}^k \subseteq (x_1, x_2^{n_2}, ..., x_{d-1}^{n_{d-1}}, x_d^{n_d}.y_d^{n_d})R + \mathrm{Ann}_R(A) \tag{5}$$

for some $k \in \mathbb{N}$. Let $A = \sum_{i=1}^{h} S_i$ be a minimal secondary representation of A. Then

$$\sqrt{\mathrm{Ann}_R(A)} = \sqrt{0 :_R \sum_{i=1}^{h} S_i} = \bigcap_{i=1}^{h} \sqrt{0 :_R S_i} = \bigcap_{\mathfrak{p} \in \mathrm{Att}(A)} \mathfrak{p}. \tag{6}$$

It goes from (5) and (6) that

$$\mathfrak{m}^k \subseteq (x_1, x_2^{n_2}, ..., x_{d-1}^{n_{d-1}}, x_d^{n_d}.y_d^{n_d})R + \bigcap_{\mathfrak{p} \in \mathrm{Att}(A)} \mathfrak{p}.$$

This implies $x_1R + (x_2^{n_2}, ..., x_{d-1}^{n_{d-1}}, x_d^{n_d}.y_d^{n_d})R \not\subseteq \bigcup_{\mathfrak{p} \in \mathrm{Att}\,(A) - \{\mathfrak{m}\}} \mathfrak{p}$. Hence, by Theorem 124 in [7], there exists $z \in (x_2^{n_2}, ..., x_{d-1}^{n_{d-1}}, x_d^{n_d}.y_d^{n_d})R$ such that $z_1 := x_1 + z \notin \bigcup_{\mathfrak{p} \in \mathrm{Att}\,(A) - \{\mathfrak{m}\}} \mathfrak{p}$. We have now z_1 is a pseudo-A-coregular. Furthermore, for each $n_1 \in \mathbb{N}$, one can find $c_{n_1} \in (x_2^{n_2}, ..., x_{d-1}^{n_{d-1}}, x_d^{n_d}.y_d^{n_d})A$ such that $z_1^{n_1} = x_1^{n_1} + c_{n_1}$. This yields the equations (3) and (4). ■

3.6. Lemma. *Let $\underline{x} = (x_1, ..., x_d)$ be an s.o.p. for A. Let $t \in \mathbb{N}$ such that $\mathfrak{m}^t \subseteq \underline{x}A + \mathrm{Ann}_R A$. Then, for any s.o.p $\underline{y} = (y_1, ..., y_d)$ of A with $x_1 = y_1, ..., x_{d-1} = y_{d-1}$ and every $\underline{n} = (n_1, ..., n_d) \in \mathbb{N}^d$, it holds*

$$I(\underline{x}(\underline{n}); A) \leqslant tI(\underline{y}(\underline{n}); A).$$

Proof: We proceed induction on d. For $d = 1$, by [6] (Lemma 2),

$$I(\underline{x}(\underline{n}); A) = \ell_R(A/x_1^{n_1}A) = \ell_R(A/(x_1^{n_1}A + \mathrm{Ann}_R A)A)$$

$$\leqslant \ell_R(A/\mathfrak{m}^{n_1 t}A) \leqslant \ell_R(A/y_1^{n_1 t}A) \leqslant t\ell_R(A/y_1^{n_1}A) = tI(\underline{y}(\underline{n}); A).$$

Assume that $d > 1$ and our assertion is true for all Artinian R-module of N-dimension smaller than d. Lemma 3.5 allow us to suppose that x_1 is an pseudo-A-coregular. Consequently, for every $n_1 \in \mathbb{N}$, $\ell_R(L/x_1^{n_1}L) < \infty$ and

$$e(x_2^{n_2}, ..., x_{d-1}^{n_{d-1}}, x_d^{n_d}; L/x_1^{n_1}L) = 0; \quad e(y_2^{n_2}, ..., y_{d-1}^{n_{d-1}}, y_d^{n_d}; L/x_1^{n_1}L) = 0.$$

Therefore,

$$e(x_1^{n_1}, x_2^{n_2}, ..., x_{d-1}^{n_{d-1}}, x_d^{n_d}; A) = e(x_2^{n_2}, ..., x_{d-1}^{n_{d-1}}, x_d^{n_d}; 0 :_A x_1^{n_1})$$

and

$$e(y_1^{n_1}, y_2^{n_2}, ..., y_{d-1}^{n_{d-1}}, y_d^{n_d}; A) = e(y_2^{n_2}, ..., y_{d-1}^{n_{d-1}}, y_d^{n_d}; 0 :_A y_1^{n_1}).$$

Hence

$$I(\underline{x}(\underline{n}); A) = I(x_2^{n_2}, ..., x_{d-1}^{n_{d-1}}, x_d^{n_d}; 0 :_A x_1^{n_1}) \tag{7}$$

and

$$I(\underline{y}(\underline{n}); A) = I(y_2^{n_2}, ..., y_{d-1}^{n_{d-1}}, y_d^{n_d}; 0 :_A y_1^{n_1}) = I(y_2^{n_2}, ..., y_d^{n_d}; 0 :_A x_1^{n_1}). \tag{8}$$

Because

$$\mathfrak{m}^k \subseteq (x_1^{n_1}, x_2^{n_2}, ..., x_d^{n_d})A + \mathrm{Ann}_R A \subseteq (x_2^{n_2}, ..., x_{d-1}^{n_{d-1}}, x_d^{n_d})A + \mathrm{Ann}_R(0 :_A x_1^{n_1}),$$

we can apply the inductive hypothesis for $0 :_A x_1^{n_1}$ to obtain

$$I(x_2^{n_2}, ..., x_{d-1}^{n_{d-1}}, x_d^{n_d}; 0 :_A x_1^{n_1}) \leqslant tI(y_2^{n_2}, ..., y_d^{n_d}; 0 :_A x_1^{n_1}). \tag{9}$$

The proposition now follows from (7), (8) and (9). ■

We now already to prove our main theorem.

Let $\underline{y} = (y_1, ..., y_d)$ be arbitrary s.o.p of A. Then we can connect $\underline{x}$ and $\underline{y}$ by a sequence of not more than $(2d + 1)$ s.o.p's of A with the property that two consecutive ones differ by at most one element. By repeated applications of Lemma 3.6, one can find natural numbers t_1, t_2 such that, $\forall \underline{n} \in \mathbb{N}^d$,

$$I(\underline{x}(\underline{n}); A) \leqslant t_1 I(\underline{y}(\underline{n}); A) \text{ and } I(\underline{y}(\underline{n}); A) \leqslant t_2 I(\underline{x}(\underline{n}); A).$$

The proof is then complete.

The above theorem means that the least degree of all polynomials bounding from above $I(\underline{x}(\underline{n}); A)$ is a numerical invariant of A. From now on, we denote this invariant by $\mathrm{ld}_R(A)$ or $\mathrm{ld}(A)$ (if there is no confusion). We stipulate that the degree of the zero-polynomial is equal to $-\infty$.

We close this section by an example in which we can easily calculate the invariant ld. Besides, it shows that the function $\ell_R(0 :_A (x_1^{n_1}, ..., x_d^{n_d})R)$ may be not a polynomial even when $n_1, ..., n_d$ large enough.

3.7. Example. Let $B = k[[Y_1, Y_2, Y_3]]/(Y_1Y_3, Y_2Y_3)$, where k is a field and We denote by x_1, x_2 the natural images of $Y_1 + Y_3, Y_2 + Y_3$ in B, then $\underline{x} = (x_1, x_2)$ forms a system of parameters for the Noetherian module B (as B-module). It can be verified that

$$\ell_B(B/(x_1^{n_1}, x_2^{n_2})B) = n_1n_2 \cdot e_0(x_1, x_2; B) + \min\{n_1, n_2\},$$

where $e_0(\underline{x}; B)$ is the Hilber-Samuel multiplicity of Noetherian B with respect to $\underline{x}$. Denote by $\mathfrak{n}$ the maximal ideal of the local ring B and E the injective hull of $B/\mathfrak{n}$. Set $B^\vee :=$ $\mathrm{Hom}_R(B; E)$, the Matlis dual of B. Then, $B^\vee$ is an Artinian B-module and $\underline{x}$ is also a system of parameters for $B^\vee$. It goes from basic facts of Matlist dual that

$$\ell_B(B/(x_1^{n_1}, x_2^{n_2})B) = \ell_B((B/(x_1^{n_1}, x_2^{n_2})B)^\vee) = \ell_B(0 :_{B^\vee} ((x_1^{n_1}, x_2^{n_2})B).$$

Hence,

$$\ell_B(0 :_{B^\vee} (x_1^{n_1}, x_2^{n_2})R) = n_1n_2 \cdot e(x_1, x_2; B) + \min\{n_1, n_2\}.$$

Moreover, because

$$\ell_B(B/(x_1, x_2)^tB) = \ell_B((B/(x_1, x_2)^tB)^\vee) = \ell_B(0 :_{B^\vee} (x_1, x_2)^tB),$$

for all $t \in \mathbb{N}$, we get $e_0(\underline{x}; B) = e(\underline{x}; B^\vee)$ by [10] (Formular 14.1 page 107) and [3] (4.4). Accordingly,

$$I(x_1^{n_1}, x_2^{n_2}); B^\vee) = \min\{n_1, n_2\}.$$

Therefore, $\mathrm{ld}(B^\vee) = 1$.

4. Connect to local homology modules

We devote this section to show some ralations between the invariant ld and local homology modules. But let us first recall the definition of local homology which is first introduced in [5].

4.1. Definition. Let I be an ideal in R and let i is a non-negative integer. Then the R-module $\varprojlim_t \mathrm{Tor}_i^R(R/I^t; A)$ is called *ith- local homology module of A with respect to I* and denoted by $H_i^I(A)$.

Denote by $\widehat{R}$ be the $\mathfrak{m}$-completion of R. As A is Artinian over R, for all $i \geqslant 0$ and $t > 0$, on can check that $\mathrm{Tor}_i^R(R/I^t; A)$ is an Artine R-module. Thus $\mathrm{Tor}_i^R(R/I^t; A)$ can be regarded as an $\widehat{R}$-module and therefore $H_i^{\mathfrak{m}}(A)$ too. It have been shown in [5] that, for all $i \geqslant 0$, $H_i^{\mathfrak{m}}(A)$ is Noetherian over $\widehat{R}$ and $H_i^{\mathfrak{m}}(A) \cong H_i^{\mathfrak{m}\widehat{R}}(A)$ as $\widehat{R}$-modules.

4.2. Lemma. *Let $s = s(A)$ be the stability index of A. Then $H_0^{\mathfrak{m}}(A) = A/\mathfrak{m}^s A$.*

Proof: $H_0^{\mathfrak{m}}(A) = \varprojlim_t \big(\mathrm{Tor}_0^R(R/\mathfrak{m}^t; A)\big) = \varprojlim_t (R/\mathfrak{m}^t \otimes_R A) = \varprojlim_t (A/\mathfrak{m}^t A) = A/\mathfrak{m}^s A.$

4.3. Lemma. *Assume that $\ell_R(H_i^{\mathfrak{m}}(A)) < \infty$ for all $i < d$. Let $k \in \mathbb{N}$ be such that $\mathfrak{m}^k H_i^{\mathfrak{m}}(A) = 0, \forall i = 0, ..., d-1$. Then there holds*

$$\ell(0 :_A \underline{x}R) - e(\underline{x}; A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_R(H_i^{\mathfrak{m}}(A))$$

for all system of parameters $\underline{x}R$ contained in $\mathfrak{m}^{k2^d}$

Proof: It suffices to prove our lemma in the case R is complete. We make induction on d. When $d = 1$, and $\mathfrak{m}^k H_0^{\mathfrak{m}}(A) = 0$ and let $\underline{x} = x_1$ is a s.o.p of A with $x_1 R \subseteq \mathfrak{m}^{2k}$. As $0 = \mathfrak{m}^k H_0^{\mathfrak{m}}(A) = \mathfrak{m}^k(A/\mathfrak{m}^s A)$, we have $\mathfrak{m}^k A \subseteq \mathfrak{m}^s A$ and thus have $k \geqslant s$ by the definition of the stability index.

$$\ell_R(0 :_A \underline{x}R) - e(\underline{x}; A) = \ell_R(A/x_1 A) \geqslant \ell_R(A/m^s A) = \ell_R\big(H_0^{\mathfrak{m}}(A)\big). \tag{10}$$

On the other hand, choosing $r \in \mathbb{N}$ such that $m^r \subseteq x_1 R + \mathrm{Ann}_R(A)$, then

$$\begin{aligned}\ell_R(0 :_A \underline{x}R) - e(\underline{x}; A) = \ell_R(A/x_1 A) &= \ell_R\Big(A/((x_1 R + \mathrm{Ann}_R(A))A)\Big) \\ &\leqslant \ell_R(A/\mathfrak{m}^r A) \leqslant \ell_R(A/\mathfrak{m}^s A) = \ell_R\big(H_0^{\mathfrak{m}}(A)\big).\end{aligned} \tag{11}$$

By (10) and (11), our assertion have proved in the case $d = 1$.

Now suppose that $d > 1$ and our statement is true for all Artine R-module of $\mathrm{N}-\dim$ smaller than d. Let $\underline{x} = (x_1, ..., x_d)$ be arbitrary s.o.p contained in $\mathfrak{m}^{k2^d}$. By Lemma 3.5, we can assume that x_1 is a pseudo-A-coregular. Let us consider two following exact sequences

$$0 \longrightarrow x_1 A \longrightarrow A \longrightarrow A/x_1 A \longrightarrow 0 \tag{12}$$

and

$$0 \longrightarrow (0 :_A x_1 R) \longrightarrow A \xrightarrow{x_1} x_1 A \longrightarrow 0. \tag{13}$$

Because $\ell_R(A/x_1 A) < \infty$ we get $H_i^{\mathfrak{m}}(A/x_1 A) = 0, \forall i > 0$. The exact sequence (12) then implies that

$$H_i^{\mathfrak{m}}(A) \cong H_i^{\mathfrak{m}}(x_1 A), \forall i = 1, ..., d-1. \tag{14}$$

By virtue of [5] (4.2), the exact sequence (13) yields the long exact sequence

$$\begin{aligned}\cdots \longrightarrow H_i^{\mathfrak{m}}(A) \longrightarrow H_{i-1}^{\mathfrak{m}}(0 :_A x_1) \longrightarrow H_{i-1}^{\mathfrak{m}}(A) \xrightarrow{x_1} H_{i-1}^{\mathfrak{m}}(A) \longrightarrow \cdots \\ \longrightarrow H_1^{\mathfrak{m}}(A) \longrightarrow H_0^{\mathfrak{m}}(0 :_A x_1) \longrightarrow H_0^{\mathfrak{m}}(A) \xrightarrow{x_1} H_0^{\mathfrak{m}}(x_1 A) \longrightarrow 0.\end{aligned} \tag{15}$$

By our assumption, $x_1 H_i^{\mathfrak{m}}(A) = 0, \forall i < d$, there is an isomorphism

$$H_0^{\mathfrak{m}}(0 :_A x_1) \cong H_0^{\mathfrak{m}}(A)$$

and for each $i \in \{1, ..., d-1\}$, there is a short exact sequence

$$0 \longrightarrow H^{\mathfrak{m}}_i(A) \longrightarrow H^{\mathfrak{m}}_{i-1}(0 :_A x_1) \longrightarrow H^{\mathfrak{m}}_{i-1}(A) \longrightarrow 0.$$

Accordingly,

$$\mathfrak{m}^{2k} H^{\mathfrak{m}}_j(0 :_A x_1) = 0, \forall j = 0, ..., d-2 \tag{16}$$

and moreover,

$$\begin{aligned} &\ell_R\big(H^{\mathfrak{m}}_{i-1}(0 :_A x_1)\big) = \ell_R\big(H^{\mathfrak{m}}_i(A)\big) + \ell_R\big(H^{\mathfrak{m}}_{i-1}(A)\big) < \infty, \ \forall i = 1, ..., d-1 \ \text{ and} \\ &\ell_R\big(H^{\mathfrak{m}}_0(0 :_A x_1)\big) = \ell_R\big(H^{\mathfrak{m}}_0(A)\big) < \infty. \end{aligned} \tag{17}$$

Since

$$(x_2, ..., x_d)R \subseteq \mathfrak{m}^{k2^d} = \mathfrak{m}^{2k.2^{d-1}},$$

(16) and (17) enable us to apply the inductive hypothesis for the s.o.p $(x_2, ..., x_d)$ of R-module $(0 :_A x_1)$ and then obtain

$$\begin{aligned} &\ell_R(0 :_{(0:x_1)} (x_2, ..., x_d)R) - e(x_2, ..., x_d; 0 :_A x_1) = \\ &\qquad \sum_{j=0}^{d-2} \binom{d-2}{j} \ell_R(H^{\mathfrak{m}}_j(0 :_A x_1)) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_R(H^{\mathfrak{m}}_i(A)). \end{aligned}$$

The inductive step completes by the observation that

$$\begin{aligned} \ell_R(0 :_A \underline{x}) - e(\underline{x}; A) &= \ell_R(0 :_{(0:_A x_1)} (x_2, ..., x_d)R) - e(x_2, ..., x_d; 0 :_A x_1) \\ &\quad + e(x_2, ..., x_d; A/x_1A) \\ &= \ell_R(0 :_{(0:_A x_1)} (x_2, ..., x_d)R) - e(x_2, ..., x_d; 0 :_A x_1) \end{aligned}$$

as $\mathrm{N}-\dim(A/x_1A) = 0$.

4.4. Lemma. *Let $\underline{x}$ be a s.o.p of A. Let $\underline{m} = (m_1, ..., m_d), \underline{n} = (n_1, ..., n_d) \in \mathbb{N}^d$ with $m_i \leqslant n_i, \forall i = 1, ..., d$. Then*

$$I(\underline{x}(\underline{m}); A) \leqslant I(\underline{x}(\underline{n}); A).$$

Proof: As usually, we can assume addition that R is complete. Moreover, because the function $I(\underline{x}(\underline{n}); L)$ is not dependent on oder of $x_1, ..., x_d$, it reduces our lemma to the case $m_1 = n_1, ..., m_{d-1} = n_{d-1}, m_d \leqslant n_d$. We do induction on d. For $d = 1$,

$$I(\underline{x}(\underline{m}); A) = \ell(A/x_1^{m_1}A) \leqslant \ell(A/x_1^{n_1}A) = I(\underline{x}(\underline{n}); A).$$

In the next step, we can apply the same method in proof of Lemma 3.6 and then comlete the inductive progress.

4.5. Corollary. *Let $\underline{x}$ be arbitrary s.o.p of A. Then there holds*

$$\ell_R(0:_A \underline{x}R) - e(\underline{x};A) \leqslant \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_R(H_i^{\mathfrak{m}}(A)).$$

Proof: If $\ell_R(H_i^{\mathfrak{m}}(A)) = +\infty$ for some $i \in \{0, ..., d-1\}$, then we have nothing to prove. When $\ell_R(H_i^{\mathfrak{m}}(A)) < \infty, \forall i < d$ we can find $k \in \mathbb{N}$ such that $\mathfrak{m}^k(H_i^{\mathfrak{m}}(A)) = 0, \forall i < d$. Taking $n_1, ..., n_d \in \mathbb{N}$ with $n_i \geqslant k2^d, \forall i = 1, ..., d$. Then

$$\begin{aligned}\ell_R(0:_A \underline{x}R) - e(\underline{x};A) &\leqslant \ell_R(0:_A (x_1^{n_1}, ..., x_d^{n_d})R) - e(x_1^{n_1}, ..., x_d^{n_d}; A) \\ &= \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_R(H_i^{\mathfrak{m}}(A))\end{aligned}$$

by Lemma 4.4 and Lemma 4.3.

4.6. Theorem. *$ld(A) = -\infty \Longleftrightarrow H_i^{\mathfrak{m}}(A) = 0, \forall i < d$.*

Proof: If $H_i^{\mathfrak{m}}(A) = 0, \forall i < d$ then it follows from Corollary 4.5 that $\mathrm{ld}(A) = -\infty$.

We prove the inverse by induction on d. For $d = 1$ and let $\underline{x} = x_1$ be a s.o.p of A. Then, because $\mathrm{ld}(A) = -\infty$, we have $\ell_R(0:_A x_1) - e(x_1; A) = 0$. By virtue of [3] (5.3), it implies x_1 is A-coregular so that

$$A = x_1^k A \subseteq \mathfrak{m}^k A \subseteq A, \forall k \in \mathbb{N}.$$

Thus $A = \mathfrak{m}^k A, \forall k \in \mathbb{N}$ and so $H_0^{\mathfrak{m}}(A) = A/\mathfrak{m}^{S(A)}A = 0$ by Lemma (4.2). Therefore our statement have proved for the case $d = 1$.

Assume that $d > 1$ and our assertion is true for all Artinian R-module of $\mathrm{N}-\dim$ smaller than d. Let $\underline{x} = (x_1, ..., x_d)$ be a s.o.p of A. As $\mathrm{ld}(A) = -\infty$, then $\ell_R(0:_A \underline{x}R) - e(\underline{x};A) = 0$. By [3] (5.3), x_1 is A-coregular. The exact sequence $0 \longrightarrow (0:_A x_1) \longrightarrow A \xrightarrow{x_1} A \longrightarrow 0$ then generates the long exact sequence

$$\begin{aligned}\cdots \longrightarrow H_i^{\mathfrak{m}}(0:_A x_1) \longrightarrow H_i^{\mathfrak{m}}(A) \xrightarrow{x_1} H_i^{\mathfrak{m}}(A) \longrightarrow H_{i-1}^{\mathfrak{m}}(0:_A x_1) \cdots \\ \longrightarrow H_0^{\mathfrak{m}}(0:_A x_1) \longrightarrow H_0^{\mathfrak{m}}(A) \xrightarrow{x_1} H_0^{\mathfrak{m}}(A) \longrightarrow 0.\end{aligned} \tag{18}$$

As x_1 is A-coregular,

$$0 = \ell_R(0:_A (\underline{x}R)) - e(\underline{x};A) = \ell_R(0:_{(0:x_1)} (x_2, ..., x_d)R) - e(x_2, ..., x_d; 0:_A x_1)$$

and thus $\mathrm{ld}(0:_A x_1) = -\infty$. Now we can apply the inductive hypothesis for $(0:_A x_1)$ to have $H_i^{\mathfrak{m}}(0:_A x_1) = 0, \forall i = 0, ..., d-2$. By this, the long exact sequence (18) gives an isomorphism

$$H_i^{\mathfrak{m}}(A) \xrightarrow{x_1} H_i^{\mathfrak{m}}(A), \forall i < d.$$

Hence, for every $i < d$, $H_i^{\mathfrak{m}}(A) = x_1 H_i^{\mathfrak{m}}(A)$ and consequently, $\forall k \in \mathbb{N}$,

$$H_i^{\mathfrak{m}}(A) = x_1^k H_i^{\mathfrak{m}}(A) \subseteq \mathfrak{m}^k H_i^{\mathfrak{m}}(A).$$

This deduces that

$$H_i^{\mathfrak{m}}(A) \subseteq \bigcap_{k \geqslant 0} \mathfrak{m}^k H_i^{\mathfrak{m}}(A) = 0, \forall i < d$$

by [5] (3.1) and the induction is finished.

Co Cohen-Macaulay modules is introduced in [17]. This class of Artinian modules is in some sense dual to the well known theory of Cohen-Macaulay modules. We are going to give a character for co Cohen- Macaulay modules in term of the invariant ld.

4.7. Corollary. *The following conditions are equivalent:*

i) there exists a s.o.p $\underline{x}$ *of* A *such that* $\ell_R(0 :_A \underline{x}A) = e(\underline{x}; A)$,

ii) $ld(A) = -\infty$,

iii) for arbitrary s.o.p $\underline{x}$ *of* A, *we have* $\ell_R(0 :_A \underline{x}A) = e(\underline{x}; A)$,

iv) there exists a s.o.p of A *which is also a* A*-cosequence,*

v) Every s.o.p of A *is also a* A*-cosequence,*

vi) A *is co Cohen-Macaulay, that is* $N - \dim_R A = Width A$,

vii) $H_i^{\mathfrak{m}}(A) = 0$, *for all* $i = 0, ..., d-1$.

Proof: The statements $(i) \Longleftrightarrow (ii)$ and $(ii) \Longleftrightarrow (iii)$ yield from the definition of ld.

$(iii) \Longleftrightarrow (vii)$ is nothing else Theorem 4.6.

$(i) \Longleftrightarrow (iv)$ and $(iii) \Longleftrightarrow (v)$ are essentially Theorem 5.3 in [[CN]].

In oder to prove $(v) \Longleftrightarrow (vi)$ we first recall that $\mathrm{Width}_R(A) \leqslant \mathrm{N} - \dim_R(A)$ by [17] (2.11). Observe that every A-cosequence is also a subset of a system of parameter of A (see [17] (2.14)). This proves $(v) \Longrightarrow (vi)$. The inverse is clear by definition of Width and the previous observation.

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