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REPRESENTATIONS OF THE REAL DIAMOND LIE GROUP

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Abstract. We present explicit formulas *representations* of the real diamond Lie algebra obtained from the normal polarization on K-orbits. From this we have list irreducible unitary representations of the real diamond Lie group that is coincide with the representations via Fedosov deformation quantisation. Here the computations are more simple for use star-product.

1. Introduction

The method of orbits discovered in the pioneering works of Kirillov [8] is a universal base for performing harmonic analysis on homogeneous spaces and for constructing new methods of integrating linear differential equations. We give the explicit form of the representations of the real diamond Lie group which is coincide the representations obtained from the deformation quantisation (see [4, 5]). The representations first appeared in the noncommutative integration method of linear differential equations as a "quantum" analogue of the noncommutative Mishchenko-Fomenko integration method for finite-dimensional Hamiltonian systems [9].

Quantum groups are *group Hopf algebras*, i.e. replace C*-algebras by special Hopf algebras "of functions". It is therefore interesting to ask whether we could describe quantum groups as some repeated extensions of some kind quantum strata of coadjoint orbits? We are attempting to give a positive answer to this question. It is not yet completely described but we obtained a reasonable answer. Let us describe the main ingredients of our approach. Let us in few words describe the structure of the paper. We introduce on K-orbits in §2, Darboux coordinates, normal polarization on K-orbits and the notion of the quantization of K-orbits in §3. Then, Hamiltonian functions in canonical coordinates of the co-adjoint orbit $\mathcal{O}_\xi$, the operators ℓ_A which define the representations (the $\xi-$representations) of the real diamond Lie algebra are constructed in §4 and finally, by exponentiating them, we obtain the corresponding unitary representations of the real diamond Lie group $\mathbb{R} \ltimes \mathbb{H}_3$.

2. The description of K-orbits

From the orbit method, it is well-known that coadjoint orbits are homogeneous symplectic manifolds with respect to the natural Kirillov structure form on coadjoint

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orbits. Let G be a real connected n-dimensional Lie group and $\mathfrak{g}$=LieG be its Lie algebra. The action of the adjoint representation Ad^* of the Lie group defines a fibration of the dual space $\mathfrak{g}^*$ into even-dimensional orbits (the K-orbits). The maximum dimension of a K-orbit is $n-r$, where $r = \mathrm{ind}\mathfrak{g}$, the index of the Lie algebra, defined as the dimension of the annihilator of a general covector. We say that a linear functional (a covector) ξ has the degeneration degree s if it belongs to a K-orbit $\mathcal{O}_\xi$ which $\dim \mathcal{O}_\xi = n - r - 2s, s = 0...., (n-r)/2$.

We decompose the space $\mathfrak{g}^*$ into a sum of nonintersecting invariant algebraic surfaces Φ_s consisting of K-orbits with the same dimension. This can be done as follows. We let α_i denote the coordinates of the covector F in the dual basis, $F = f_i e^i$ with $\langle e^i, e_j\rangle = \delta^i_j$, where $\{e_j\}$ is the basis of $\mathfrak{g}$. The vector fields on $\mathfrak{g}^*$

$$\eta_i(F) \equiv C_{ij}(F)\frac{\partial}{\partial f_j}, \quad C_{ij}(F) \equiv C^k_{ij} f_k$$

are generators of the transformation group G acting on the space $\mathfrak{g}^*$, and their linear span therefore constitutes the space $T_F\mathcal{O}_\xi$ tangent to the orbit $\mathcal{O}_\xi$ running through the point F. Thus, the dimension of the orbit $\mathcal{O}_\xi$ is determined by the rank of the matrix C_{ij},

$$\dim\mathcal{O}_\xi = \mathrm{rank} C_{ij}(\xi).$$

It can be easily verified that the rank of C_{ij} is constant over the orbit. Therefore, equating the corresponding minors of $C_{ij}(X)$ to zero and "forbidding" the vanishing of lower-order minors, we obtain polynomial equations that define a surface Φ_s,

$$\begin{aligned}
\Phi_0 &= \{F \in \mathfrak{g}^* \mid \neg(\Delta^1(F) = 0)\};\\
\Phi_s &= \{F \in \mathfrak{g}^* \mid \Delta^s(F) = 0,\ \overline{(\Delta^{s+1}(F) = 0)}\},\ s = 1, \dots, \frac{n-r}{2} - 1;\\
\Phi_{\frac{n-r}{2}} &= \{F \in \mathfrak{g}^* \mid \Delta^{\frac{n-r}{2}}(F) = 0\}.
\end{aligned}$$

Here, by $\Delta^s(F)$ we denote the collection of all minors of $C_{ij}(F)$ of the size $n - r - 2s + 2$, the condition $\Delta^s(F) = 0$ indicates that all the minors of $C_{ij}(F)$ of the size $n - r - 2s + 2$ vanish at the point F, and $\overline{(\Delta^s(F) = 0)}$ means that the corresponding minors do not vanish simultaneously at F.

Note that in the general case, the surface Φ_s consists of several nonintersecting invariant components, which we distinguish with subscripts as $\Phi_s = \Phi_{sa} \cup \Phi_{sb} \dots$. (To avoid stipulating each time that the space Φ_s is not connected, we assume the convention that s in parentheses, (s), denotes a specific type of the orbit with the degeneration degree s.) Each component $\Phi_{(s)}$ is defined by the corresponding set of homogeneous polynomials $\Delta^{(s)}_\theta(F)$ satisfying the conditions

$$\eta_i \Delta^{(s)}_\theta(F)\big|_{\Delta^{(s)}(F)=0} = 0.$$

Although the invariant algebraic surfaces $\Phi_{(s)}$ are not linear spaces, they are star sets, i.e., $F \in \Phi_{(s)}$, implies $tF \in \Phi_{(s)}$ for $t \in \mathbb{R}^1$.

3. Darboux coordinates and normal polarization

By ω_ξ we denote the Kirillov form on the orbit $\mathcal{O}_\xi$. It defines a symplectic structure and acts on the vectors a and b tangent to the orbit as

$$\omega_\xi(a,b) = \langle \xi, [\alpha,\beta] \rangle,$$

where $a = ad^*_\alpha \xi$ and $b = ad^*_\beta \xi$. The restriction of Poisson brackets to the orbit coincides with the Poisson bracket generated by the symplectic form ω_ξ. According to the well-known Darboux theorem (see [arnold]), there exist local canonical coordinates (Darboux coordinates) on the orbit $\mathcal{O}_\xi$ such that the form ω_ξ becomes

$$\omega_\xi = dp_k \wedge dq^k; \quad k = 1,\dots,\frac{1}{2}\dim O_\xi = \frac{n-r}{2} - s,$$

where s is the degeneration degree of the orbit. Let be $F \in \mathcal{O}_\xi, F = f_i e^i$. It can be easily seen that the trasition to canoniccal Darboux coordinates $(f_i) \mapsto (p_k, q^k)$ amounts to constructing analytic functions $f_i = f_i(q,p,\xi)$ of variables (p,q) satisfying the conditions

$$f_i(0,0,\xi) = \xi_i;$$

$$\frac{\partial f_i(q,p,\xi)}{\partial p_k}\frac{\partial f_j(q,p,\xi)}{\partial q^k} - \frac{\partial f_j(q,p,\xi)}{\partial p_k}\frac{\partial f_i(q,p,\xi)}{\partial q^k} = C^l_{ij} f_l(q,p,\xi).$$

We choose the the canonical Darboux coordinates with impulse p's-coordinates. From this we can deduce that the Kirillov form ω_ξ locally are canonical and every element $A \in \mathfrak{g} = \text{Lie}G$ can be considered as a function $\tilde{A}$ on $\mathcal{O}_\xi$, linear on p's-coordinates, i.e.

There exists on each coadjoint orbit a local canonical system of Darboux coordinates, in which the Hamiltonian function $\tilde{A} = a_i(q,p,\xi)e^i$, $A \in \mathfrak{g}$, are linear on p's impulsion coordinates and in theses coordinates,

$$a_i(q,p,\xi) = \alpha^k_i(q) p_k + \chi_i(q,\xi); \quad \text{rank}\alpha^k_i(q) = \frac{1}{2}\text{dim}\mathcal{O}_\xi. \tag{1}$$

We have

Theorem 3.1. [2]) *The linear transition to canonica coordinates on the orbit $\mathcal{O}_\xi$ exists if and only if there exists a normal polarization (in general, complex) associated with the linear functional ξ, i.e., a subalgebra $\mathfrak{h} \subset \mathfrak{g}^c$ such that*

$$\dim \mathfrak{h} = n - \frac{1}{2}\dim \mathcal{O}_\xi, \quad \langle \xi, [\mathfrak{h},\mathfrak{h}] \rangle = 0, \quad \xi + \mathfrak{h}^\perp \subset \mathcal{O}_\xi.$$

In the classical method of orbits, the polarization appears as an $(n - \frac{1}{2}\dim \mathcal{O}_\xi)$-dimensional subalgebra $\mathfrak{h} \subset \mathfrak{g}^c$, with its one-dimensional representation determined by the functional ξ. In our case, a normal polarization determines linear transition (1) to the canonical coordinates.

It can be easily seen that relacing the functional ξ with another covector belonging to the same orbit leads to replacing the polarization $\mathfrak{h}$ with the cojugate one $\tilde{\mathfrak{h}}$, with the Darboux coordinates corresponding to these two polarizations related by a point transformation, $\tilde{q^k} = \tilde{q^k}(q); \tilde{p_k} = \frac{\partial q^l}{\tilde{q^k}} p_l$. Therefore, the choice of a specific representative of the orbit is not essential. On the other hand, if the polarizations are not conjugate, the coresponding Darboux coordinates are related by a more general canonical transformation. With the "quantum" canonical transformation determined (with q and p being operators, see below), we can thus construct the intertwining operator between the two representations obtained via the method of orbits involving two polarizations. In the case where no polarization exists for a given functional, the transition to Darboux coordinates (which is nonlinear in the p variables) can still be constructed, and the representation of $\mathfrak{g}$ can still be defined; this representation is the basis for the harmonic analysis on Lie groups and homogeneous spaces (applications of the method of orbits to harmonic analysis go beyond the scope of this paper and are not considered here). In other words, the existence of a polarization is a useful property but is not necessary for the applicability of the method of orbits.

We define the notion of the quantization of K-orbits. We now view the transition functions $f_i(q, p; \xi)$ to local canonical coordinates as symbols of operators that are defined as follows: the variables p_k are replaced with derivatives, $p_k \to \hat{p}_k \equiv -i\hbar\frac{\partial}{\partial q^k}$, and the coordinates of a covector f_i become the linear operators

$$f_i(q, p; \xi) \to \hat{f}_i\left(q, -i\hbar\frac{\partial}{\partial q}; \xi\right) \tag{2}$$

(with $\hbar$ being a positive real parameter). We require that the operators $\hat{f}_i$ satisfy the commutation relations

$$\frac{i}{\hbar}[\hat{f}_i, \hat{f}_j] = C^l_{ij}\hat{f}_l.$$

If the transition to the canonical coordinates is linear, i.e., a normal polarization exists for orbits of a given type, it is obvious that

$$\hat{f}_i = -i\hbar\alpha^k_i(q)\frac{\partial}{\partial q^k} + \chi_i(q, \xi) \tag{3}$$

With Hamiltonian function $\tilde{A} = a_i(q, p, \xi)e^i$, $A \in \mathfrak{g}$, the operators $\hat{a}_i$ as shown by evidence.

We note that in the "classical" limit as $\hbar \to 0$, we have the commutator of linear operators goes into the Poisson bracket on the coalgebra,

$$\frac{i}{\hbar}[\cdot, \cdot] \to \{\cdot, \cdot\}.$$

We introduce the operators

$$\ell_k(q, \partial_q) \equiv \frac{i}{\hbar}\hat{a}_k(q, p; \xi). \tag{4}$$

It is obvious that
$$[\ell_i, \ell_j] = C_{ij}^k \ell_k$$

Definition 3.2. *Let $f_i = f_i(q, p; \xi)$ be a transition to canonical coordinates on the orbit $\mathcal{O}_\xi$ of the Lie algebra $\mathfrak{g}$. The operators $\ell_i(q, \partial_q)$ is called the representation (the ξ-representation) of the Lie algebra $\mathfrak{g}$.*

4. The real diamond Lie group

The real diamond Lie group has a lot of nontrivial 2-dimensional coadjoint orbits, which are the half-planes, the hyperbolic cylinders and the hyperbolic paraboloids (see [11]). We should find out explicit formulas for each of these orbits. Our main result is find the complete list of irreducible unitary representations of this group.

4.1 Preliminary results

The so called real diamond Lie algebra is the 4-dimensional solvable Lie algebra $\mathfrak{g}$ with basis X, Y, Z, T satisfying the following commutation relations, see [10]:

$$[X, Y] = Z, [T, X] = -X, [T, Y] = Y,$$

$$[Z, X] = [Z, Y] = [T, Z] = 0.$$

These relations show that this real diamond Lie algebra $\mathfrak{g} = \mathbb{R} \ltimes \mathfrak{h}_3$ is an extension of the one-dimensional Lie algebra $\mathbb{R}T$ by the Heisenberg algebra $\mathfrak{h}_3$ with basis X, Y, Z, where the action of T on Heisenberg algebra $\mathfrak{h}_3$ is defined by the matrix

$$ad_T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The real diamond Lie algebra is isomorphic to $\mathbb{R}^4$ as vector spaces. The coordinates in this standard basis is denote by (a, b, c, d). We identify its dual vector space $(\mathbb{R} \ltimes \mathfrak{h}_3)^*$ with $\mathbb{R}^4$ with the help of the dual basis X^*, Y^*, Z^*, T^* and with the local coordinates as $(\alpha, \beta, \gamma, \delta)$. Thus, the general form of an element of $\mathfrak{g}$ is $A = aX + bY + cZ + dT$ and the general form of an element of $(\mathbb{R} \ltimes \mathfrak{h}_3)^*$ is $\xi = \alpha X^* + \beta Y^* + \gamma Z^* + \delta T^*$. The co-adjoint action of the real diamond group $\mathrm{G} = \mathbb{R} \ltimes \mathbb{H}_3$ on $(\mathbb{R} \ltimes \mathfrak{h}_3)^*$ is given by

$$\langle K(g)\xi, A\rangle = \langle \xi, \mathrm{Ad}(g^{-1})A\rangle, \quad \forall \xi \in (\mathbb{R} \ltimes \mathfrak{h}_3)^*, g \in \mathbb{R} \ltimes \mathbb{H}_3 \text{ and } A \in \mathbb{R} \ltimes \mathfrak{h}_3.$$

Fixing $\xi \in (\mathbb{R} \ltimes \mathfrak{h}_3)^*$ and denote the co-adjoint orbit of $\mathbb{R} \ltimes \mathbb{H}_3$ in $\mathbb{R} \ltimes \mathfrak{h}_3$, passing through ξ by
$$\mathcal{O}_\xi = K(G)\xi := \{K(g)\xi \quad | g \in \mathbb{R} \ltimes \mathbb{H}_3\}.$$

By a direct computation one obtains (see 2, 7, also see [11]:

- Each point of the line $\alpha = \beta = \gamma = 0$ is a 0-dimensional co-adjoint orbit

$$\mathcal{O}^1 = \mathcal{O}_{(0,0,0,\delta)}.$$

- The set $\alpha \neq 0, \beta = \gamma = 0$ is union of 2-dimensional co-adjoint orbits, which are just the *half-planes*

$$\mathcal{O}^2 = \{(x, 0, 0, t) \quad | \quad x, t \in \mathbb{R}, \alpha x > 0\}.$$

- The set $\alpha = \gamma = 0, \beta \neq 0$ is a union of 2-dimensional co-adjoint orbits, which are *half-planes*

$$\mathcal{O}^3 = \{(0, y, 0, t) \quad | \quad y, t \in \mathbb{R}, \beta y > 0\}.$$

- The set $\alpha\beta \neq 0, \gamma = 0$ is decomposed into a family of 2-dimensional co-adjoint orbits, which are *hyperbolic cylinders*

$$\mathcal{O}^4 = \{(x, y, 0, t) \quad | x, y, t \in \mathbb{R} \quad \& \quad \alpha x > 0, \beta y > 0, xy = \alpha\beta\}.$$

- The open set$\gamma \neq 0$ is decomposed into a family of 2-dimensional co-adjoint orbits, which are just the *hyperbolic paraboloids*

$$\mathcal{O}^5 = \{(x, y, \gamma, t) \quad | x, y, t \in \mathbb{R} \quad \& \quad xy - \alpha\beta = \gamma(t - \delta)\}.$$

Thus,

$$\mathcal{O}_\xi = \mathcal{O}^1 \cup \mathcal{O}^2 \cup \mathcal{O}^3 \cup \mathcal{O}^4 \cup \mathcal{O}^5.$$

4.2. Hamiltonian functions in canonical coordinates of the orbits $\mathcal{O}_\xi$

Each element $A \in \mathfrak{g}$ can be considered as the restriction of the corresponding linear functional $\tilde{A}$ onto co-adjoint orbits, considered as a subset of g^*, $\tilde{A}(\xi) = \langle \xi, A \rangle$. It is well-known that this function is just the Hamiltonnian function, associated with the Hamiltonian vector field ξ_A, defined by the formula

$$(\xi_A f)(x) := \frac{d}{dt} f(x \exp(tA))|_{t=0}, \forall f \in C^\infty(\mathcal{O}_\xi).$$

It is well-known the relation $\xi_A(f) = \{\tilde{A}, f\}, \forall f \in C^\infty(\mathcal{O}_\xi)$. Denote by ψ the symplectomorphism from $\mathbb{R}^2$ onto $\mathcal{O}_\xi$

$$(p, q) \in \mathbb{R}^2 \mapsto \psi(p, q) \in \mathcal{O}_\xi.$$

Then we have:

Proposition 4.1. *1. Hamiltonian function $\tilde{A}$ in canonical coordinates (p, q) of the orbit $\mathcal{O}_\xi$ is of the form*

$$\tilde{A} \circ \psi(p, q) = \begin{cases} dp + a\alpha e^{-q}, & on \quad \mathcal{O}^2 \\ dp + b\beta e^{q}, & on \quad \mathcal{O}^3 \\ dp + a\alpha e^{-q} + b\beta e^{q}, & on \quad \mathcal{O}^4 \\ (d \pm b\gamma e^{q})p \pm ae^{-q} \pm b(\alpha\beta - \gamma\delta)e^{q} + c\gamma, & on \quad \mathcal{O}^5. \end{cases}$$

2. In the canonical coordinates (p, q) of the orbit $\mathcal{O}_\xi$, the Kirillov form ω_ξ is coincided with the standard form $dp \wedge dq$.

Proof.

1. We adapt the diffeomorphism ψ to each of the following cases (for 2-dimensional co-adjoint orbits, only)

- With $\alpha \neq 0, \beta = \gamma = 0$

$$(p, q) \in \mathbb{R}^2 \mapsto \psi(p, q) = (\alpha e^{-q}, 0, 0, p) \in \mathcal{O}^2.$$

Element $\xi \in \mathfrak{g}^*$ is of the form $\xi = \alpha X^* + \beta Y^* + \gamma Z^* + \delta T^*$, hence the value of the function $f_A = \tilde{A}$ on the element $A = aX + bY + cZ + dT$ is $\tilde{A}(\xi) = \langle F, A\rangle =$

$$\langle \alpha X^* + \beta Y^* + \gamma Z^* + \delta T^*, aX + bY + cZ + dT\rangle = \alpha a + \beta b + \gamma c + \delta d.$$

It follows that

$$\tilde{A} \circ \psi(p, q) = a\alpha e^{-q} + dp, \tag{5}$$

- With $\alpha = \gamma = 0, \beta \neq 0$,

$$(p, q) \in \mathbb{R}^2 \mapsto \psi(p, q) = (0, \beta e^{q}, 0, p) \in \mathcal{O}^3.$$

$\tilde{A}(F) = \langle \xi, A\rangle = \alpha a + \beta b + \gamma c + \delta d$. From this,

$$\tilde{A} \circ \psi(p, q) = b\beta e^{q} + dp. \tag{6}$$

- With $\alpha\beta \neq 0, \gamma = 0$,

$$(p, q) \in \mathbb{R}^2 \mapsto \psi(p, q) = (\alpha e^{-q}, \beta e^{q}, 0, p) \in \mathcal{O}^4.$$

$$\tilde{A} \circ \psi(p, q) = a\alpha e^{-q} + b\beta e^{q} + dp. \tag{7}$$

- At last, if $\gamma \neq 0$, we consider the orbit with the first coordinate $x > 0$

$$(p, q) \in \mathbb{R}^2 \mapsto \psi(p, q) = (e^{-q}, (\alpha\beta + \gamma p - \gamma\delta)e^{q}, \gamma, p) \in \mathcal{O}^5.$$

We have

$$\begin{aligned} \tilde{A} \circ \psi(p, q) &= ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^{q} + c\gamma + dp \\ &= (d + b\gamma e^{q})p + ae^{-q} + b(\alpha\beta - \gamma\delta)e^{q} + c\gamma. \end{aligned} \tag{8}$$

The case $x < 0$ is similarly treated:

$$\begin{aligned}(p,q) \in \mathbb{R}^2 \mapsto \psi(p,q) &= (-e^{-q}, -(\alpha\beta + \gamma p - \gamma\delta)e^q, \gamma, p) \in \mathcal{O}^5.\\ \tilde{A} \circ \psi(p,q) &= -ae^{-q} - b(\alpha\beta + \gamma p - \gamma\delta)e^q + c\gamma + dp\\ &= (d - b\gamma e^q)p - ae^{-q} - b(\alpha\beta - \gamma\delta)e^q + c\gamma. \qquad (9)\end{aligned}$$

2. By a direct computation, we conclude that in the canonical coordinates the Kirillov form is the standard symplectic form $\omega = dp \wedge dq$.

The proposition is therefore proved. □

4.3. Representations of the real diamond group

Theorem 4.2. *For each $A \in \mathbb{R} \ltimes \mathfrak{h}_3$, representations of the real diamond algebra are*

$$\ell^A = \begin{cases} \ell_2^A = \big(d\partial_q + ia\alpha e^{-q}\big)\\ \ell_3^A = \big(d\partial_q + ib\beta e^q\big)\\ \ell_4^A = \big(d\partial_q + i[a\alpha e^{-q} + b\beta e^q]\big)\\ \ell_5^A = \Big((d + b\gamma e^q)\partial_q + i[ae^{-q} + b(\alpha\beta - \gamma\delta)e^q + c\gamma]\Big)\\ \ell_{5'}^A = \Big((d - b\gamma e^q)\partial_q + i[-ae^{-q} - b(\alpha\beta - \gamma\delta)e^q + c\gamma]\Big)\end{cases}$$

Proof. Applying directly (3), (4) we have:

1. If $\tilde{A} = dp + a\alpha e^{-q}$ then $\hat{A} = -i\hbar d\partial q + a\alpha e^{-q}$ and from this,

$$\ell_2^A(q, \partial_q) = \frac{i}{\hbar}[-i\hbar d\partial q + a\alpha e^{-q}] = d\partial_q + \frac{i}{\hbar}a\alpha e^{-q}$$

2. If $\tilde{A} = dp + b\beta e^q$ then $\hat{A} = -i\hbar d\partial q + b\beta e^q$ and from this,

$$\ell_3^A(q, \partial_q) = \frac{i}{\hbar}[-i\hbar d\partial q + b\beta e^q] = d\partial_q + \frac{i}{\hbar}b\beta e^q$$

3. If $\tilde{A} = dp + a\alpha e^{-q} + b\beta e^q$ then $\hat{A} = -i\hbar d\partial q + a\alpha e^{-q} + b\beta e^q$ and from this,

$$\ell_4^A(q, \partial_q) = \frac{i}{\hbar}[-i\hbar d\partial q + a\alpha e^{-q} + b\beta e^q] = d\partial_q + \frac{i}{\hbar}(a\alpha e^{-q} + b\beta e^q)$$

4. At last, if $\tilde{A} = (d \pm b\gamma e^q)p \pm ae^{-q} \pm b(\alpha\beta - \gamma\delta)e^q + c\gamma$ we obtain

$$\begin{aligned}\ell_5^A &= \Big((d + b\gamma e^q)\partial_q + i[ae^{-q} + b(\alpha\beta - \gamma\delta)e^q + c\gamma]\Big)\\ \text{or}\quad \ell_5^A &= \Big((d - b\gamma e^q)\partial_q + i[-ae^{-q} - b(\alpha\beta - \gamma\delta)e^q + c\gamma]\Big)\end{aligned}$$

The theorem is therefore proved. □

As $\mathbb{R} \ltimes \mathbb{H}_3$ is connected and simply connected Lie group, we obtain

Corollary 4.3. *The irreducible unitary representations $\mathcal{T}$ of the real diamond Lie group $\mathbb{R} \ltimes \mathbb{H}_3$ defined by*

$$\mathcal{T}(\exp A) := \exp(\ell^A); \quad A \in \mathbb{R} \ltimes \mathfrak{h}_3$$

More detail,

$$\mathcal{T}(\exp A) = \begin{cases} \exp(d\partial_q + ia\alpha e^{-q}) & \text{if } \tilde{A} \text{ is defined by (4)} \\ \exp(d\partial_q + ib\beta e^{q}) & \text{if } \tilde{A} \text{ is defined by (5)} \\ \exp(d\partial_q + i[a\alpha e^{-q} + b\beta e^{q}]) & \text{if } \tilde{A} \text{ is defined by (6)} \\ \exp((d + b\gamma e^{q})\partial_q + i[ae^{-q} + b(\alpha\beta - \gamma\delta)e^{q} + c\gamma]) & \text{if } \tilde{A} \text{ is defined by (7)} \\ \exp((d - b\gamma e^{q})\partial_q + i[-ae^{-q} - b(\alpha\beta - \gamma\delta)e^{q} + c\gamma]) & \text{if } \tilde{A} \text{ is defined by (8)} \end{cases}$$

This means that we find list all the irreducible unitary representations $\mathcal{T}(\exp A)$ of the real diamond Lie group $\mathbb{R} \ltimes \mathbb{H}_3$ that is coincide the representations via Fedosov deformation quantisation. What we did here gives us more simplisity computations in this case for use the star-product (see [6], [4], [5], [7]).

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