VNU. JOURNAL OF SCIENCE, Mathematics - Physics. T.XXI, $N_0$3 - 2005

REMARKS ON LOCAL DIMENSION OF FRACTAL MEASURE ASSOCIATED WITH THE (0, 1, 9) - PROBLEM

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Abstract. Let X be random variable taking values $0, 1, a$ with equal probability $1/3$ and let $X_1, X_2, ...$ be a sequence of independent identically distributed (i.i.d) random variables with the same distribution as X. Let μ be the probability measure induced by $S = \sum_{i=1}^{\infty} 3^{-i} X_i$. Let $\alpha(s)$ (resp. $\underline{\alpha}(s), \overline{\alpha}(s)$) denote the local dimension (resp. lower, upper local dimension) of $s \in \text{supp } \mu$.

Put

$$\overline{\alpha} = \sup\{\overline{\alpha}(s) : s \in \text{supp } \mu\}; \underline{\alpha} = \inf\{\underline{\alpha}(s) : s \in \text{supp } \mu\};$$
$$E = \{\alpha : \alpha(s) = \alpha \text{ for some } s \in \text{supp } \mu\}.$$

When $a \equiv 0 \pmod 3$, the probability measure μ is singular and it is conjectured that for $a = 3k$ (for any $k \in \mathbb{N}$), the local dimension is still the same as the case $k = 1, 2$. It means $E = [1 - \frac{\log(a)}{b \log 3}, 1]$, for a, b depend on k. Our result shows that for $k = 3$ $(a = 9), \overline{\alpha} = 1$, $\underline{\alpha} = 2/3$ and $E = [\frac{2}{3}, 1]$.

1. Introduction

By a *Probabilistic system* we mean a sequence $X_1, X_2, ...$ of i.i.d random variables with the same distribution as X, where X is a random variable taking values $a_1, a_2, ..., a_m$ with probability $p_1, p_2, ..., p_m$, respectively.

Let $S = \sum_{i=1}^{\infty} \rho^i X_i$, for $0 < \rho < 1$, then the probability measure μ induced by S, i.e.,

$$\mu(A) = \text{Prob}\{\omega : S(\omega) \in A\}$$

is called the *Fractal measure* associated with the probabilistic system.

It is specified two interesting cases.

The first case is when $m = 2, p_1 = p_2 = 1/2$ and $a_1 = 0, a_2 = 1$. In this case the fractal measure μ is known as "Infinite Bernoulli Convolutions". This measure has been studied for over sixty years but is still only partial understood today.

The second case of interest is when $m = 3, \rho = p_1 = p_2 = p_3 = 1/3$ and $a_1 = 0, a_2 = 1, a_3 = a$. Some authors have found $\overline{\alpha}, \underline{\alpha}$ and E for some concrete values of a for which μ is singular. But for each of them, the way to find $\overline{\alpha}, \underline{\alpha}$ and E is quite different. It is conjectured that the way to find them in the general case is dificult.

Let us recall that for $s \in \text{supp } \mu$ the *local dimension* $\alpha(s)$ of μ at s is defined by

$$\alpha(s) = \lim_{h \to 0^+} \frac{\log \mu(B_h(s))}{\log h}, \tag{1}$$

provided that the limit exists, where $B_h(s)$ denotes the ball centered at s with radius h. If the limit (1) does not exist, we define the upper and lower local dimension, denoted $\overline{\alpha}(s)$ and $\underline{\alpha}(s)$, by taking the upper and lower limits, respectively.

Denote

$$\overline{\alpha} = \sup\{\overline{\alpha}(s) : s \in \text{supp } \mu\} \ ; \ \underline{\alpha} = \inf\{\underline{\alpha}(s) : s \in \text{supp } \mu\},$$

and let

$$E = \{\alpha : \alpha(s) = \alpha \text{ for some } s \in \text{supp } \mu\}$$

be the attainable values of $\alpha(s)$, i.e., the range of function α definning in the supp μ.

In this paper, we consider the interest second case with $a = 9$ ($a \equiv 0 \pmod 3$). Our result is the following.

Main Theorem. For $a = 9$, $\overline{\alpha} = 1, \underline{\alpha} = \frac{2}{3}$ and $E = [\frac{2}{3}, 1]$.

The paper is organized as follows. In Section 2, we establish some auxiliary results are used to prove the formula for calculating the local dimension. In Section 3, we prove the maximal sequences, it is used to find the lower local dimension. The proof of the Main Theorem will be given in the last section.

2. The formula for calculating the local dimension

Let $X_1, X_2, \dots$ be a sequence of i.i.d random variables each taking values $0, 1, 9$ with equal probability 1/3. Let $S = \sum_{i=1}^{\infty} 3^{-i} X_i$, $S_n = \sum_{i=1}^{n} 3^{-i} X_i$ be the n-partial sum of S, and let μ, μ_n be the probability measures induced by S, S_n, respectively. For any $s = \sum_{i=1}^{\infty} 3^{-i} x_i \in \text{supp } \mu, x_i \in$ D: $= \{0, 1, 9\}$, let $s_n = \sum_{i=1}^{n} 3^{-i} x_i$ be its n-partial sum.

Let

$$\langle s_n \rangle = \{(x_1, x_2, ..., x_n) \in D^n : \sum_{i=1}^{n} 3^{-i} x_i = s_n\}.$$

Then we have

$$\mu_n(s_n) = \#\langle s_n \rangle 3^{-n} \text{ for every } n, \tag{2}$$

where $\#X$ denotes the cardinality of set X.

Two sequences $(x_1, x_2, ..., x_n)$ and $(x'_1, x'_2, ..., x'_n)$ in D^n are said to be *equivalent,* denoted by $(x_1, x_2, \dots, x_n) \approx (x'_1, x'_2, \dots, x'_n)$ if $\sum_{i=1}^{n} 3^{-i} x_i = \sum_{i=1}^{n} 3^{-i} x'_i$. We have

2.1. Claim. (i) For any $(x_1, x_2, \dots, x_n)$, $(x'_1, x'_2, \dots, x'_n)$ in D^n and $s_n = \sum_{i=1}^{n} 3^{-i} x_i$, $s'_n = \sum_{i=1}^{n} 3^{-i} x'_i$, we have $|s_n - s'_n| = k 3^{-n}$ for some $k \in \mathbb{N}$ and $x_n - x'_n \equiv k \pmod 3$.
(ii) Let $s_n < s'_n < s''_n$ be three arbitrary consecutive points in supp μ_n. Then either $s'_n - s_n$ or $s''_n - s'_n$ is not $\frac{i}{3^n}$ for $i = 1, 2, 3$ and for every $n \in N$.
(iii) Let $s_n, s'_n \in$ supp μ_n and $s_n - s'_n = \frac{1}{3^n}$ or $\frac{4}{3^n}$. Then $s_n = s_{n-1} + \frac{1}{3^n}$ is the unique representation of s_n through points in supp μ_{n-1} and $s'_n = s_{n-1} + \frac{0}{3^n}$ or $s'_n = s'_{n-1} + \frac{9}{3^n}$ or both of them, where s_{n-1}, s'_{n-1} in supp μ_{n-1}.
(iv) For any $s_n, s'_n \in$ supp μ_n such that $s_n - s'_n = \frac{2}{3^n}$. Then $s'_n = s'_{n-1} + \frac{1}{3^n}$ is the unique representation of s'_n through points in supp μ_{n-1} and $s_n = s_{n-1} + \frac{0}{3^n}$ or $s_n = s''_{n-1} + \frac{9}{3^n}$ or both of them, where $s_{n-1}, s'_{n-1}, s''_{n-1}$ in supp μ_{n-1}.
Proof. It is proved similarly as proof of Claim 2.1, 2.2 in [11].

2.2. Corollary. (i) Let $s_{n+1} \in$ supp μ_{n+1} and $s_{n+1} = s_n + \frac{1}{3^{n+1}}, s_n \in$ supp μ_n. We have

$$\#\langle s_{n+1} \rangle = \#\langle s_n \rangle, \text{for every } n \geqslant 1.$$

(ii) For any $s_n, s'_n \in$ supp μ_n, if $s_n - s'_n = \frac{1}{3^n}$ or $s_n - s'_n = \frac{2}{3^n}$, then s_n, s'_n are two consecutive points in supp μ_n.
(iii) Let $s_n, s'_n \in$ supp μ_n, if $s_n - s'_n = \frac{1}{3^n}$ then $\#\langle s_n \rangle \leqslant \#\langle s'_n \rangle$.
Proof. It follows directly from Claim 2.1.

2.3. Lemma. *For any $s_n, s'_n \in$ supp μ_n, if $s_n - s'_n = \frac{3}{3^n}$, then either both of s_n, s'_n have two representations through points in supp μ_{n-1} or $s_n = s_{n-1} + \frac{x_n}{3^n}, s'_n = s'_{n-1} + \frac{x_n}{3^n}$, for $x_n \in D, s_{n-1}, s'_{n-1}$ in supp μ_{n-1}.*
Proof. Assume on the contrary, then there are the following cases.

Case 1. If $s_n = s_{n-1} + \frac{0}{3^n} = s''_{n-1} + \frac{9}{3^n}, s'_n = s'_{n-1} + \frac{0}{3^n}$ (1).
Then $s'_{n-1} - s''_{n-1} = \frac{2}{3^{n-1}}$. By Claim 2.1 (iv) $s''_{n-1} = s''_{n-2} + \frac{1}{3^{n-1}}$. It implies $s'_n = s^*_{n-1} + \frac{9}{3^n}$, where $s^*_{n-1} = s''_{n-2} + \frac{0}{3^{n-1}}$, a contradiction to (1).

Case 2. If $s_n = s_{n-1} + \frac{0}{3^n} = s''_{n-1} + \frac{9}{3^n}, s'_n = s'_{n-1} + \frac{9}{3^n}$ (2).
Then $s_{n-1} - s'_{n-1} = \frac{4}{3^{n-1}}$. By Claim 2.1 (iii) $s_{n-1} = s_{n-2} + \frac{1}{3^{n-1}}$. It implies $s'_n = s^*_{n-1} + \frac{0}{3^n}$, where $s^*_{n-1} = s_{n-2} + \frac{0}{3^{n-1}}$, a contradiction to (2).

Case 3. If $s_n = s_{n-1} + \frac{9}{3^n}, s'_n = s'_{n-1} + \frac{0}{3^n} = s''_{n-1} + \frac{9}{3^n}$ (3).
Then $s_{n-1} - s''_{n-1} = \frac{1}{3^{n-1}}$. By Claim 2.1 (iii) $s_{n-1} = s_{n-2} + \frac{1}{3^{n-1}}$. If $s'_{n-1} = s'_{n-2} + \frac{0}{3^n}$, then $s_n = s^*_{n-1} + \frac{0}{3^n}$, where $s^*_{n-1} = s'_{n-2} + \frac{1}{3^{n-1}}$, a contradiction to (3). Hence $s'_{n-1} = s''_{n-2} + \frac{9}{3^n}$. It implies $s_{n-2} - s''_{n-2} = \frac{2}{3^{n-2}}$.

If $s_{n-2} = s_{n-3} + \frac{0}{3^{n-2}}$, then there is $s'_{n-2} = s_{n-3} + \frac{1}{3^{n-1}}$. Hence $s_{n-2} = s'_{n-3} + \frac{9}{3^{n-2}}$. Thus, by repeating this argument then there are two points $s_1, s'_1 \in$ supp μ_1, such that $s_1 - s'_1 = \frac{2}{3}$, a contradiction.

Case 4. If $s_n = s_{n-1} + \frac{0}{3^n}, s'_n = s'_{n-1} + \frac{0}{3^n} = s''_{n-1} + \frac{9}{3^n}$ (4).

Then $s_{n-1}-s''_{n-1}=\frac{4}{3^{n-1}}$. By Claim 2.1 (iii) $s_{n-1}=s_{n-2}+\frac{1}{3^{n-1}}$. If $s''_{n-1}=s'_{n-2}+\frac{0}{3^n}$, then $s_n=s^*_{n-1}+\frac{9}{3^n}$, where $s^*_{n-1}=s'_{n-2}+\frac{1}{3^{n-1}}$, a contradiction to (4). Hence $s''_{n-1}=s''_{n-2}+\frac{9}{3^n}$ is the unique representation of s''_{n-1} (5). Then we have $s_{n-2}-s''_{n-2}=\frac{4}{3^{n-2}}$, by Claim 2.1 (iii) $s_{n-2}=s_{n-3}+\frac{1}{3^{n-2}}$. Then there is $s'_{n-2}=s_{n-3}+\frac{0}{3^{n-1}}$. It implies $s''_{n-1}=s'_{n-2}+\frac{0}{3^n}$ a contradiction to (5). So this case does not happen.
Observe that, from Case 3 and Case 4 there are not the cases

$$s_n=s_{n-1}+\frac{9}{3^n}, s'_n=s'_{n-1}+\frac{0}{3^n}$$

and

$$s_n=s_{n-1}+\frac{0}{3^n}, s'_n=s'_{n-1}+\frac{9}{3^n}.$$

The lemma is proved.

2.4. Corollary. (i) Let $s_n, s'_n\in$ supp μ_n, if $s_n-s'_n=\frac{3}{3^n}$, then $\#\langle s_n\rangle\leqslant\#\langle s'_n\rangle$.
(ii) For any $s_n<s'_n<s''_n$ are three consecutive points in supp μ_n and $s''_n-s'_n=\frac{3}{3^n,}$ then $s'_n-s_n\neq\frac{2}{3^n}$.
Proof. (i) It follows directly from Lemma 2.3 and Corollary 2.2(iii).
(ii) It follows directly from Lemma 2.3 and Claim 2.1 (ii).

2.5. Lemma. *Let $s_n, s'_n\in$ supp μ_n, if $s_n-s'_n=\frac{1}{3^n,}$ then*

$$\frac{\mu_n(s'_n)}{\mu_n(s_n)}\leqslant\frac{n+1}{2}.$$

Proof. We will prove the inequality by induction. Clearly the inequality holds for $n=1$. Suppose that it is true for all $n\leqslant k-1$. We consider the case $n=k$.

By Claim 2.1 (iii) and Corollary 2.2 (i), we have $\#\langle s_k\rangle=\#\langle s_{k-1}\rangle$, where

$$s_k=s_{k-1}+\frac{1}{3^k}, s_{k-1}\in \text{supp }\mu_{k-1}.$$

We consider the following cases.

Case 1. If $s'_k=s_{k-1}+\frac{0}{3^k}$ is the unique representation of s'_k through point in supp μ_{k-1}. Then $\#\langle s'_k\rangle=\#\langle s_{k-1}\rangle$. Therefore

$$\frac{\#\langle s'_k\rangle}{\#\langle s_k\rangle}=\frac{\#\langle s_{k-1}\rangle}{\#\langle s_{k-1}\rangle}=1\leqslant\frac{k+1}{2}.$$

Case 2. If s'_k has two representations through points in supp μ_{k-1},

$$s'_k=s_{k-1}+\frac{0}{3^k}=s'_{k-1}+\frac{9}{3^k}.$$

Then $s_{k-1} - s'_{k-1} = \frac{3}{3^{k-1}}$. By Lemma 2.3, either

$$s_{k-1} = s_{k-2} + \frac{x_k}{3^{k-1}} \text{ and } s'_{k-1} = s'_{k-2} + \frac{x_k}{3^{k-1}}, \ (1)$$

or

$$s_{k-1} = s_{k-2} + \frac{0}{3^{k-1}} = s''_{k-2} + \frac{9}{3^{k-1}}$$

and

$$s'_{k-1} = s'_{k-2} + \frac{0}{3^{k-1}} = s'''_{k-2} + \frac{9}{3^{k-1}}. \ (2)$$

If (1) happens, then

$$\#\langle s_{k-1}\rangle = \#\langle s_{k-2}\rangle, \#\langle s'_{k-1}\rangle = \#\langle s'_{k-2}\rangle \text{ and } s_{k-2} - s'_{k-2} = \frac{1}{3^{k-2}}.$$

By inductive hypothesis, we have

$$\frac{\#\langle s'_k\rangle}{\#\langle s_k\rangle} = \frac{\#\langle s_{k-1}\rangle + \#\langle s'_{k-1}\rangle}{\#\langle s_{k-1}\rangle} = \frac{\#\langle s_{k-2}\rangle + \#\langle s'_{k-2}\rangle}{\#\langle s_{k-2}\rangle} \leqslant 1 + \frac{k-1}{2} = \frac{k+1}{2}.$$

If (2) happens, then

$$\#\langle s_{k-1}\rangle = \#\langle s_{k-2}\rangle + \#\langle s''_{k-2}\rangle,$$

$$\#\langle s'_{k-1}\rangle = \#\langle s'_{k-2}\rangle + \#\langle s'''_{k-2}\rangle.$$

By inductive hypothesis, we have

$$\begin{aligned}\frac{\#\langle s'_k\rangle}{\#\langle s_k\rangle} &= \frac{\#\langle s_{k-1}\rangle + \#\langle s'_{k-1}\rangle}{\#\langle s_{k-1}\rangle} = 1 + \frac{\#\langle s'_{k-2}\rangle + \#\langle s'''_{k-2}\rangle}{\#\langle s_{k-2}\rangle + \#\langle s''_{k-2}\rangle} \\ &\leqslant 1 + \frac{\frac{k-1}{2}[\#\langle s_{k-2}\rangle + \#\langle s''_{k-2}\rangle]}{\#\langle s_{k-2}\rangle + \#\langle s''_{k-2}\rangle} \leqslant 1 + \frac{k-1}{2} = \frac{k+1}{2}. \qquad (3)\end{aligned}$$

The lemma is proved.

2.6. Lemma. *Let $s_n, s'_n \in$ supp μ_n, and $s_n - s'_n = \frac{3}{3^n}$. We always have*

$$\frac{\mu_n(s'_n)}{\mu_n(s_n)} \leqslant n.$$

Proof. Since $s_n - s'_n = \frac{3}{3^n}$, by Lemma 2.3, we have two following cases.

Case 1. Both of s_n, s'_n have the unique representations through points s_{n-1}, s'_{n-1} in supp μ_{n-1}, respectively. Then

$$\#\langle s_n\rangle = \#\langle s_{n-1}\rangle, \#\langle s'_n\rangle = \#\langle s'_{n-1}\rangle \text{ and } s_{n-1} - s'_{n-1} = \frac{1}{3^{n-1}}.$$

By Lemma 2.5, we have

$$\frac{\#\langle s'_n\rangle}{\#\langle s_n\rangle} = \frac{\#\langle s'_{n-1}\rangle}{\#\langle s_{n-1}\rangle} \leqslant \frac{(n-1)+1}{2} < n.$$

Case 2. Both of s_n, s'_n has two representations through points in supp μ_{n-1},

$$s_n = s_{n-1} + \frac{0}{3^n} = s''_{n-1} + \frac{9}{3^n}, s'_n = s'_{n-1} + \frac{0}{3^n} = s'''_{n-1} + \frac{9}{3^n}.$$

Then $s_{n-1} - s'_{n-1} = s''_{n-1} - s'''_{n-1} = \frac{1}{3^{n-1}}$. By Lemma 2.5, we have

$$\#\langle s'_{n-1}\rangle \leqslant \frac{(n-1)+1}{2}\#\langle s_{n-1}\rangle, \#\langle s'''_{n-1}\rangle \leqslant \frac{(n-1)+1}{2}\#\langle s''_{n-1}\rangle.$$

Hence, we have

$$\begin{aligned} \frac{\#\langle s'_n\rangle}{\#\langle s_n\rangle} &= \frac{\#\langle s'''_{n-1}\rangle + \#\langle s'_{n-1}\rangle}{\#\langle s_{n-1}\rangle + \#\langle s''_{n-1}\rangle} \\ &\leqslant \frac{\frac{(n-1)+1}{2}[\#\langle s_{n-1}\rangle + \#\langle s''_{n-1}\rangle]}{\#\langle s_{n-1}\rangle + \#\langle s''_{n-1}\rangle} \\ &\leqslant \frac{n}{2} < n. \end{aligned} \tag{4}$$

The lemma is proved.

Using Lemma 2.5, 2.6 we will prove the following lemma, which is used to establish a useful formula for calculating the local dimension.

2.7. Lemma. *For any two consecutive points s_n and s'_n in supp μ_n, we have*

$$\frac{\mu_n(s_n)}{\mu_n(s'_n)} \leqslant n.$$

Proof. By (2) it is sufficient to show that $\frac{\#\langle s_n\rangle}{\#\langle s'_n\rangle} \leqslant n$. We will prove the inequality by induction. Clearly the inequality holds for $n = 1$. Suppose that it is true for all $n \leqslant k$. Let $s_{k+1} > s'_{k+1}$ be two arbitrary consecutive points in supp μ_{k+1}. Write

$$s_{k+1} = s_k + \frac{x_{k+1}}{3^{k+1}},\ s_k \in \text{supp } \mu_k, x_{k+1} \in D.$$

We consider the following cases for x_{k+1}

Case 1. If $x_{k+1} = 1$, then $s_{k+1} = s_k + \frac{1}{3^{k+1}}$. By Corollary 2.2 (i) $\#\langle s_{k+1}\rangle = \#\langle s_k\rangle$. We have $s'_{k+1} = s_k + \frac{0}{3^{k+1}}$. Assume that s'_{k+1} has an other representation

$$s'_{k+1} = s'_k + \frac{9}{3^{k+1}}, s'_k \in \text{supp } \mu_k.$$

Then $\#\langle s'_{k+1}\rangle \leqslant \#\langle s_k\rangle+\#\langle s'_k\rangle$ and $s_k-s'_k=\frac{3}{3^k}$. By Lemma 2.5, we have $\#\langle s'_k\rangle \leqslant k\#\langle s_k\rangle$. Thus,

$$\frac{\#\langle s'_{k+1}\rangle}{\#\langle s_{k+1}\rangle} \leqslant \frac{\#\langle s_k\rangle+\#\langle s'_k\rangle}{\#\langle s_k\rangle} \leqslant \frac{(1+k)\#\langle s_k\rangle}{\#\langle s_k\rangle} = 1+k.$$

Case 2. If $x_{k+1}=0$, then $s_{k+1}=s_k+\frac{0}{3^{k+1}}$.

a) If s_{k+1} has not any other representation. Then $\#\langle s_{k+1}\rangle = \#\langle s_k\rangle$.

For any $s^*_{k+1}=s^*_k+\frac{x^*_{k+1}}{3^{k+1}} < s_{k+1}=s_k$. It implies $s^*_k < s_k$.

Let $s'_k \in$ supp μ_k be the biggest value smaller than s_k then $s'_k < s_k$ are two consecutive points in supp μ_k and $s_k - s'_k \neq \frac{3}{3^k}$. Then there are three following cases.

If $s_k = s'_k + \frac{1}{3^k}$ then $s'_{k+1} = s'_k + \frac{1}{3^{k+1}}$. By Corollary 2.2 (i) $\#\langle s'_{k+1}\rangle = \#\langle s'_k\rangle$. Therefore

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} = \frac{\#\langle s_k\rangle}{\#\langle s'_k\rangle} \leqslant k < k+1.$$

The case $s_k = s'_k + \frac{2}{3^k}$ is proved similarly to the case $s_k = s'_k + \frac{1}{3^k}$.

If $s_k - s'_k > \frac{3}{3^k}$, then $s'_{k+1} = s'_k + \frac{9}{3^{k+1}}$ is the unique representation of s'_{k+1}. Hence, $\#\langle s'_{k+1}\rangle = \#\langle s'_k\rangle$.

Therefore

$$\frac{\#\langle s'_{k+1}\rangle}{\#\langle s_{k+1}\rangle} = \frac{\#\langle s'_k\rangle}{\#\langle s_k\rangle} \leqslant k < k+1.$$

b) If s_{k+1} has an other representation $s_{k+1} = s'_k + \frac{9}{3^{k+1}}$.

b1) If s'_k, s_k are two consecutive points in supp μ_k, then let $s''_k < s'_k$ in supp μ_k are two consecutive points. By Corollary 2.4 (ii) we have two cases

$$s'_k - s''_k > \frac{3}{3^k} \text{ or } s'_k - s''_k = \frac{1}{3^k}.$$

If $s'_k - s''_k > \frac{3}{3^k}$, then $s'_{k+1} = s'_k + \frac{1}{3^{k+1}}$. By Corollary 2.2 (i) $\#\langle s'_{k+1}\rangle = \#\langle s'_k\rangle$. Therefore

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} = \frac{\#\langle s'_k\rangle+\#\langle s_k\rangle}{\#\langle s'_k\rangle} \leqslant 1+k.$$

If $s'_k - s''_k = \frac{1}{3^k}$, then $\#\langle s'_k\rangle \leqslant \#\langle s''_k\rangle$ and $s'_{k+1} = s''_k + \frac{9}{3^{k+1}}$ is the unique representation of s'_{k+1}. Hence, $\#\langle s'_{k+1}\rangle = \#\langle s''_k\rangle$. Therefore

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} = \frac{\#\langle s'_k\rangle+\#\langle s_k\rangle}{\#\langle s''_k\rangle} \leqslant 1+\frac{\#\langle s_k\rangle}{\#\langle s''_k\rangle} \leqslant 1+\frac{\#\langle s_k\rangle}{\#\langle s'_k\rangle} \leqslant 1+k.$$

b2) Assume that there is s^*_k in supp μ_k and $s^*_k \in (s'_k, s_k)$. Then there are two cases. If $s_k - s^*_k = \frac{1}{3^k}$, then $s'_{k+1} = s^*_k + \frac{1}{3^{k+1}}$, so $\#\langle s'_{k+1}\rangle = \#\langle s^*_k\rangle$ and $\#\langle s_k\rangle \leqslant \#\langle s^*_k\rangle$. By Corollary 2.2 (ii), s'_k, s^*_k, s_k are three consecutive points in supp μ_k. Therefore

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s'_{k+1}\rangle} = \frac{\#\langle s'_k\rangle+\#\langle s_k\rangle}{\#\langle s^*_k\rangle} \leqslant \frac{\#\langle s'_k\rangle}{\#\langle s_k\rangle} + \frac{\#\langle s_k\rangle}{\#\langle s_k\rangle} \leqslant k+1.$$

If $s_k^* - s_k' = \frac{1}{3^k}$, then $s_{k+1}' = s_k^* + \frac{1}{3^{k+1}}$. So $\#\langle s_{k+1}'\rangle = \#\langle s_k^*\rangle$. Since $s_k^* - s_k' = \frac{1}{3^k}$, by Claim 2.1 (iii) $s_k^* = s_{k-1}^* + \frac{1}{3^k}$. Since $s_k - s_k' = \frac{3}{3^k}$, by Lemma 2.3 we have following two cases. If both of s_k', s_k have two representations through points in supp μ_{k-1},

$$s_k = s_{k-1} + \frac{0}{3^k} = s_{k-1}' + \frac{9}{3^k},$$
$$s_k' = s_{k-1}^* + \frac{0}{3^k} = s_{k-1}'' + \frac{9}{3^k}.$$

Then $s_{k-1} - s_{k-1}^* = s_{k-1}' - s_{k-1}'' = \frac{1}{3^{k-1}}$.

Hence, by Corollary 2.2 (iii),

$$\#\langle s_k'\rangle = \#\langle s_{k-1}''\rangle + \#\langle s_{k-1}^*\rangle \geqslant \#\langle s_{k-1}\rangle + \#\langle s_{k-1}'\rangle = \#\langle s_k\rangle.$$

Therefore, by Lemma 2.5

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s_{k+1}'\rangle} = \frac{\#\langle s_k'\rangle + \#\langle s_k\rangle}{\#\langle s_k^*\rangle} \leqslant \frac{2\#\langle s_k'\rangle}{\#\langle s_k^*\rangle} \leqslant 2\frac{k+1}{2} = k+1.$$

If both of s_k', s_k have the unique representations through points in supp μ_{k-1}.

Since $s_k^* - s_k = \frac{1}{3^k}, s_k^* = s_{k-1}^* + \frac{1}{3^k}$, we have $s_k = s_{k-1}^* + \frac{0}{3^k}$. By Lemma 2.3, $s_k = s_{k-1} + \frac{0}{3^k}$. It implies $s_{k-1} - s_{k-1}^* = \frac{1}{3^{k-1}}$. Hence,

$$\#\langle s_k\rangle = \#\langle s_{k-1}\rangle, \#\langle s_k'\rangle = \#\langle s_{k-1}^*\rangle = \#\langle s_k^*\rangle$$

and s_{k-1}^*, s_{k-1} are two consecutive points. Therefore

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s_{k+1}'\rangle} = \frac{\#\langle s_k'\rangle + \#\langle s_k\rangle}{\#\langle s_k^*\rangle} = \frac{\#\langle s_{k-1}^*\rangle + \#\langle s_{k-1}\rangle}{\#\langle s_{k-1}^*\rangle} \leqslant 1 + (k-1) < k+1.$$

Case 3. If $x_{k+1} = 9$. We assume that $s_{k+1} = s_k + \frac{9}{3^{k+1}}$ is the unique representation of s_{k+1} through points in supp μ_k. Then $\#\langle s_{k+1}\rangle = \#\langle s_k\rangle$.

Let $s_k' \in$ supp μ_k be the smallest value bigger than s_k then $s_k < s_k'$ are two consecutive points in supp μ_k. Since $s_{k+1} = s_k + \frac{9}{3^{k+1}}$ is the unique representation of s_{k+1}, so we have following two cases.

a) If $s_k' = s_k + \frac{1}{3^k}$ or $s_k' = s_k + \frac{2}{3^k}$. Then $s_{k+1}' = s_k' + \frac{1}{3^{k+1}}$. So $\#\langle s_{k+1}'\rangle = \#\langle s_k'\rangle$. Therefore

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s_{k+1}'\rangle} = \frac{\#\langle s_k\rangle}{\#\langle s_k'\rangle} \leqslant k < k+1.$$

b) $s_k' > s_k + \frac{3}{3^k}$. Then we have $s_{k+1}' = s_k'' + \frac{9}{3^{k+1}}$ is the unique representation of s_{k+1} through point in supp μ_k, where $s_k'' < s_k$ are two consecutive points in supp μ_k.

Then

$$\frac{\#\langle s_{k+1}\rangle}{\#\langle s_{k+1}'\rangle} = \frac{\#\langle s_k\rangle}{\#\langle s_k''\rangle} \leqslant k < k+1.$$

The lemma is proved.

The following proposition provides a useful formula for calculating the local dimension and it is proved similarly as the proof of Proposition 2.3 in [11] and using Lemma 2.7.

2.8. Proposition. *For $s \in$ supp μ, we have*

$$\alpha(s) = \lim_{n\to\infty} \frac{|\log \mu_n(s_n)|}{n \log 3},$$

provided that the limit exists. Otherwise, by taking the upper and lower limits respectively we get the formulas for $\overline{\alpha}(s)$ and $\underline{\alpha}(s)$.

3. The maximal sequence

For each infinite sequence $x = (x_1, x_2, \dots) \in D^\infty$ defines a point $s \in$ supp μ by

$$s = S(x) := \sum_{n=1}^{\infty} 3^{-n} x_n.$$

By Proposition 2.8 the lower local dimension (respestively, the upper local dimension) will be determined by an element $x = (x_1, x_2, \dots) \in D^\infty$ for which $\#\langle(x_1, x_2, \dots)\rangle$ has the largest value (respestively, the smallest value). This suggests the following definition.

3.1. Definition. We say that $x(n) = (x_1, x_2, \dots, x_n) \in D^n$, for every $n \in N$, is a *maximal sequence* (respestively, *minimal sequence*) if $\#\langle y(n)\rangle \leqslant \#\langle x(n)\rangle$ (respestively, $\#\langle y(n)\rangle \geqslant \#\langle x(n)\rangle$) for every $y(n) = (y_1, y_2, \dots, y_n) \in D^n$.

3.2. Corollary. If $x = (x_1, x_2, \dots) \in D^\infty$ satisfying $x(n) = (x_1, x_2, \dots, x_n) \in D^n$ is a maximal sequence (respestively, minimal sequence) for every $n \in N$, then $\overline{\alpha} = \alpha(s)$, (respestively, $\underline{\alpha} = \alpha(s)$), where $s = \sum_{n=1}^{\infty} 3^{-n} x_n$.

Note that $x(n) = (x_1, x_2, \dots, x_n) = (0, 0, \dots, 0); x(n) = (1, 1, \dots, 1)$ or $x(n) = (9, 9, \dots, 9)$, we have $\#\langle x(n)\rangle = 1$ for every $n \in N$. So they are minimal sequences in D^n. Hence by Proposition 2.8 and Corollary 3.2, we have $\overline{\alpha} = \alpha(s) = 1$, where $s = \sum_{i=1}^{\infty} 3^{-i} x_i$. Thus, we only need consider the maximal sequences.

We denote

$$\langle x(k)\rangle = \{(y_1, \dots, y_k) \in D^k : (y_1, \dots, y_k) \approx (x_1, \dots, x_k)\}$$

where $x(k) = (x_1, \dots, x_k)$. We called $x(n) = (x_1, x_2, \dots, x_n) \in D^n$ a *mutiple sequence* if $\#\langle x(n)\rangle > 1$. Otherwise, $x(n)$ is called a *prime sequence.*

3.3. Claim. Let $x(k) = (x_1, \dots, x_k) \in D^k$. Then $x(k)$ is a mutiple sequence if and only if it contains $(1, a, 0)$ or $(0, a, 9)$, for any $a \in D$.

Proof. It is easy to see the proof of this claim.

By Claim 3.3, we call each element in the set $\{(1, a, 0), (0, a, 9)\}$, for any $a \in D$, a *generator.*

3.4. Claim. Let $x(n) = (x_1, \dots, x_k, 0, 0, 1, 1, x_{k+5}, \dots, x_n) \in D^n$, then

$$\#\langle x(n)\rangle = \#\langle (x_1, \dots, x_k, 0, 0)\rangle \ \#\langle (1, 1, x_{k+5}, \dots, x_n)\rangle.$$

Proof. Clearly that $\#\langle x(n)\rangle \geqslant \#\langle (x_1, \dots, x_k, 0, 0)\rangle \ \#\langle (1, 1, x_{k+5}, \dots, x_n)\rangle$. Assume that $\#\langle x(n)\rangle > \#\langle (x_1, \dots, x_k, 0, 0)\rangle \ \#\langle (1, 1, x_{k+5}, \dots, x_n)\rangle$. Then there is $x'(n) = (x'_1, \dots, x'_n) \in D^n$ such that:

1. $(x'_1, \dots, x'_{k+2}) \approx (x_1, \dots, x_k, 0, 0); (x'_{k+3}, \dots, x'_n) \approx (1, 1, x_{k+5}, \dots, x_n)$.
2. $(x'_{k+1}, \dots, x'_{k+4})$ is a mutiple sequence.

By Claim 3.3, then $(x'_{k+1}, \dots, x'_{k+4})$ must contain a sequence $(1, a, 0)$ or $(0, a, 9)$, for $a \in D$. Without loss of the generality, we assume that it contains $(0, a, 9)$. Then we consider the following cases.

Case 1. $(x'_{k+1}, x'_{k+2}, x'_{k+3}) = (0, a, 9)$, then $(1, 1, x_{k+5}, \dots, x_n) \approx (9, x'_{k+4}, \dots, x'_n)$. It implies

$$\begin{aligned} 8 &= \left|\frac{x'_{k+4} - 1}{3} + \frac{x'_{k+5} - x_{k+5}}{3^2} + \dots + \frac{x'_n - x_n}{3^n}\right| \\ &\leqslant 9\left|\frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n}\right| < 9.\frac{1}{2} = \frac{9}{2}, \end{aligned}$$

a contradiction.

Case 2. $(x'_{k+2}, x'_{k+3}, x'_{k+4}) = (0, a, 9)$, then $(1, 1, x_{k+5}, \dots, x_n) \approx (a, 9, x'_{k+5}, \dots, x'_n)$, for $a = 1$ or $a = 9$. From case 1 we have a contradiction. For $a = 0$, we have

$$1 - 0 + \frac{1 - 9}{3} + \frac{x_{k+5} - x'_{k+5}}{3^2} + \dots + \frac{x_n - x'_n}{3^n} = 0.$$

It implies

$$\begin{aligned} \frac{5}{3} &= \left|\frac{x_{k+5} - x'_{k+5}}{3^2} + \dots + \frac{x_n - x'_n}{3^n}\right| \\ &\leqslant 9\left|\frac{1}{3^2} + \dots + \frac{1}{3^n}\right| < 9.\frac{1}{6} = \frac{3}{2}, \end{aligned}$$

a contradiction.

Therefore

$$\#\langle x(n)\rangle = \#\langle (x_1, \dots, x_k, 0, 0)\rangle \#\langle (1, 1, x_{k+5}, \dots, x_n)\rangle.$$

The claim is proved.

3.5. Claim. Let $x(n) = (1, \dots, 1, 0, 0) \in D^n$. Put $H_n = \#\langle x(n)\rangle$. Then

$$H_3 = 2, H_n = H_{n-1} + [\frac{n}{2}], \text{for n} \geqslant 3.$$

Therefore

$$H_n = \begin{cases} \frac{n^2-1}{4} & \text{if } n \text{ is odd;} \\ \frac{n^2}{4} & \text{if } n \text{ is even,} \end{cases}$$

where $[x]$ denotes the largest integer $\leqslant x$.
Proof. We will prove by induction. Clearly that the claim is true for n=3. Assume that it holds for all $n \leqslant k$. We consider the case $n = k+1$.

We put $s_n = \sum_{i=1}^{n} 3^{-i} x_i$. We have $s_{k+1} = s'_k + \frac{9}{3^k} = s_k + \frac{0}{3^k}$ and

$$s_k = s'_{k-2} + \frac{1}{3^{k-1}} + \frac{9}{3^k} = s_{k-2} + \frac{1}{3^{k-1}} + \frac{0}{3^k},$$

where $s_{k-2} = \langle (1, \ldots, 1) \rangle$. Therefore

$$H_{k+1} = H_k + \#\langle s_k \rangle$$

and

$$\#\langle s_k \rangle = \#\langle s'_{k-2} \rangle + \#\langle s_{k-2} \rangle = \#\langle s'_{k-2} \rangle + 1.$$

By inductive hypothesis, we have $\#\langle s'_{k-2} \rangle = [\frac{k-1}{2}]$. On the other hand, it is easy to see that

$$[\frac{k-1}{2}] + 1 = [\frac{k+1}{2}].$$

Thus, $H_{k+1} = H_k + [\frac{k+1}{2}]$.

Then by considering the cases n is odd or n if even, we have the last resul in the claim.

The claim is proved.

3.6. Claim. Let $x(n) = (x_1, \ldots, x_k, 0, 1, \ldots, 1, 0, 0) \in D^n$, then

$$\#\langle x(n) \rangle = \#\langle (x_1, \ldots, x_k, 0) \rangle \ \#\langle (\underbrace{1, \ldots, 1, 0, 0}_{n-k-1}) \rangle$$

or

$$\#\langle x(n) \rangle \leqslant ([\frac{n-k-1}{2}] + \frac{1}{2} \#\langle (\underbrace{1, \ldots, 1, 0, 0}_{n-k-2}) \rangle) \ \#\langle s_{k+2} \rangle,$$

where s_{k+2} is some point in supp μ_{k+2}.
Proof. Put $s_n = \sum_{i=1}^{n} 3^{-i} x_i, \ m = n-k-1$. We will show that

$$\#\langle s_n \rangle = H_m \#\langle s_{k+1} \rangle$$

and

$$\#\langle s_n \rangle \leqslant ([\frac{m}{2}] + \frac{1}{2} H_{m-1}) \#\langle s'_{k+2} \rangle.$$

We have

$$\#\langle s_n \rangle = H_{m-1} \#\langle s_{k+2} \rangle + [\frac{m}{2}] \#\langle s'_{k+2} \rangle,$$

where $s_{k+2} - s'_{k+2} = \frac{1}{3^{k+2}}, s_{k+2}, s'_{k+2} \in$ supp μ_{k+2}. By Claim 2.1 (ii) $s_{k+2} = s_{k+1} + \frac{1}{3^{k+2}}$, for s_{k+1} is some point in supp μ_{k+1}. It implies $\#\langle s_{k+2} \rangle = \#\langle s_{k+1} \rangle$.

If $s'_{k+2} = s_{k+1} + \frac{0}{3^{k+2}}$ is the unique representation of s'_{k+2}, then $\#\langle s'_{k+2}\rangle = \#\langle s_{k+1}\rangle$. Hence

$$\begin{aligned}\#\langle s_n\rangle &= H_{m-1}\#\langle s_{k+2}\rangle + [\frac{m}{2}]\#\langle s'_{k+2}\rangle \\ &= (H_{m-1} + [\frac{m}{2}])\#\langle s_{k+1}\rangle \\ &= H_m\#\langle s_{k+1}\rangle. \end{aligned} \tag{5}$$

If $s'_{k+2} = s_{k+1} + \frac{0}{3^{k+2}} = s'_{k+1} + \frac{9}{3^{k+2}}$. Then $s_{k+1} - s'_{k+1} = \frac{3}{3^{k+1}}$. By Corollary 2.4 (i) $\#\langle s'_{k+1}\rangle \geqslant \#\langle s_{k+1}\rangle$. It implies $\#\langle s'_{k+2}\rangle \geqslant 2\#\langle s_{k+1}\rangle = 2\#\langle s_{k+2}\rangle$. Therefore

$$\begin{aligned}\#\langle s_n\rangle &= H_{m-1}\#\langle s_{k+2}\rangle + [\frac{m}{2}]\#\langle s'_{k+2}\rangle \\ &\leqslant (\frac{1}{2}H_{m-1} + [\frac{m}{2}])\#\langle s'_{k+2}\rangle. \end{aligned} \tag{6}$$

The claim is proved.

3.7. Proposition. Let

$$\begin{aligned} x^0 &= (\underbrace{1,1,1,1,0,0},\underbrace{1,1,1,1,0,0}\dots) \\ x^1 &= (1,\underbrace{1,1,1,1,0,0},\underbrace{1,1,1,1,0,0},\dots) \\ x^2 &= (1,1,\underbrace{1,1,1,1,0,0},\underbrace{1,1,1,1,0,0},\dots) \\ x^3 &= (1,1,0,\underbrace{1,1,1,1,0,0},\underbrace{1,1,1,1,0,0},\dots) \\ x^4 &= (1,1,0,0,\underbrace{1,1,1,1,0,0},\underbrace{1,1,1,1,0,0},\dots) \\ x^5 &= (1,1,1,0,0,\underbrace{1,1,1,1,0,0},\underbrace{1,1,1,1,0,0},\dots) \end{aligned} \tag{7}$$

are six sequenses in D^∞. Put $F_{6n+i} = \#\langle x^i_{6n+i}\rangle$ for $i = 0,1,2,3,4,5,\ n \in N$. Then we have

(i) $F_{6n} = 9.9^{n-1}$; $F_{6n+1} = 12.9^{n-1}$; $F_{6n+2} = 16.9^{n-1}$
$F_{6n+3} = 24.9^{n-1}$; $F_{6n+4} = 36.9^{n-1}$; $F_{6n+2} = 54.9^{n-1}$

(ii) $\#\langle t_{6n+i}\rangle \leqslant F_{6n+i}$, for $i = 0,1,2,3,4,5$ and for any $n \in N$, where t_{6n+i} is an arbitrary point in supp μ_{6n+i}.

Proof. (i) It is easy to check that $\#\langle(1,1,0,1,1,1,1,0,0)\rangle = 24$. Hence from Claim 3.5 and Claim 3.4, we get the claim (i).

(ii) We will prove the claim by induction. It is straightforward to check that the assertion holds for $n = 1$. Suppose that it is true for all $n \leqslant k (k \geqslant 1)$. We show that the proposition is true for $n = k+1$. Let $t(6(k+1)+i)$ be an arbitrary point in supp $\mu_{6(k+1)+i}$.

At first we prove for the case $i = 0$.

Write $t(6k+6) = (t(6k+2), y_3, y_4, y_5, y_6)$. We consider the following cases

Case 1. If (y_4, y_5, y_6) is not a generator, then $\#\langle t_{6n+6}\rangle = \#\langle t_{6n+5}\rangle \leqslant F_{6n+5} \leqslant F_{6n+6}$.

Case 2. If (y_4, y_5, y_6) is a generator.

Without loss of generality, we may assume that $(y_4, y_5, y_6) = (1, a, 0)$.

2.1. If $a = 1$ then

$$\langle t(6k+6)\rangle = \langle (t(6k+2), y_3, 1, 1, 0)\rangle \cup \langle (t(6k+2), y_3, 0, 1, 9)\rangle.$$

Hence

$$\begin{aligned}\#\langle t(6k+6)\rangle &= \#\langle t(6k+3)\rangle + \#\langle t(6k+4)\rangle \\ &\leqslant F_{6k+3} + F_{6k+4} \\ &\leqslant F_{6k+6}. \end{aligned} \tag{8}$$

2.2. If $a \neq 1$. Without loss of generality, we assume that $a = 0$.

If $y_3 \neq 1$. Then $(y_3, y_4, 0)$ is not a generator. So the result is similar to the case $a = 1$.

If $y_3 = 1$ then

$$\begin{aligned}\langle t(6k+6)\rangle &= \langle (t(6k+2), 1, 1, 0, 0)\rangle \cup \langle (t(6k+2), 1, 0, 0, 9)\rangle \\ &\cup \langle (t(6k+2), 0, 1, 9, 0)\rangle \cup \langle (t(6k+2), 0, 0, 9, 9)\rangle .tag9\end{aligned}$$

Then by replating when $y_2 \neq 1$ or $y_2 = 1$ and go on, we have two cases **2.2.1.** If $(t(6k+2)) = (1, \dots, 1)$. Then by claim 3.5, we have $\#\langle t(6k+6)\rangle \leqslant F_{6k+6}$.

2.2.2. Let $(t(6k+2)) = (x_1, \dots, x_l, 0, 1, 1, \dots 1, 0, 0)$. Then by Claim 3.6 and by putting $m = n - l - 1$, $m \geqslant 4$. We have

$$\#\langle t(6k+6)\rangle = \#\langle t(l+1)\rangle H_m$$

or

$$\#\langle t(6k+6)\rangle \leqslant ([\frac{m}{2}] + \frac{1}{2} H_{m-1}) \#\langle s_{l+2}\rangle.$$

Clearly for $m = 4, 5, 6$ we have

$$\#\langle t(l+1)\rangle H_m \leqslant F_{6k+6}$$

and

$$([\frac{m}{2}] + \frac{1}{2} H_{m-1}) \#\langle s_{l+2}\rangle \leqslant F_{6k+6}.$$

For $m \geqslant 6$. Write $m = 6t + j,\ t \geqslant 1, j = 0, 1, 2, 3, 4, 5$.

For $j = 0, m = 6t, l + 1 = [(6k+6) - (6t) - 1] + 1 = 6(k - t + 1)$. By Claim 3.5, we have

$$\begin{aligned} H_m = H_{6t} = 9t^2,\ F_{6(k-t+1)} = 9.9^{k-t}, \\ H_{m-1} = H_{6t-1} = 9t^2 - 3t,\ F_{6(k-t+1)+1} = 12.9^{k-t}. \end{aligned}$$

Hence

$$\#\langle t(l+1)\rangle H_m \leqslant F_{6(k-t+1)}H_m \leqslant 9.9^{k-t}.9t^2 \leqslant F_{6(k+1)},$$

and

$$\begin{aligned}([\frac{m}{2}] + \frac{1}{2}H_{m-1})\#\langle s_{l+2}\rangle &\leqslant (3t + \frac{9t^2-3t}{2})(12.9^{k-t})\\ &= 6(9t^2+3t)9^{k-t} \leqslant F_{6(k+1)}.tag10\end{aligned}$$

Thus, $\#\langle t(6(k+1))\rangle \leqslant F_{6(k+1)}$.

Similarly for $j = 1,2,3,4,5$, we get $\#\langle t(6(k+1))\rangle \leqslant F_{6(k+1)}$.

By repeating this caculus for $i = 1,3,4,5$ we have $\#\langle t(6(k+1)+i)\rangle \leqslant F_{6(k+1)+i}$.

Now we consider case $i = 2$. We will show that $\#\langle t(6(k+1)+2)\rangle \leqslant F_{6(k+1)+2}$. By similar argument as above, we have

$$\#\langle t(6(k+1)+2)\rangle = \#\langle (t(6k+4), y_1, y_2, y_3, y_4)\rangle \leqslant F_{6(k+1)+2}$$

if $(y_1, y_2, y_3, y_4) \neq (1,1,0,0)$.

So we will show that $\#\langle (t(6k+4),1,1,0,0)\rangle \leqslant F_{6(k+1)+2}$. **1.** If $y_{6k+4} \neq 1$, without loss of generality we may assume that $y_{6k+4} = 0$. Then by using notes as above, we have

$$\begin{aligned} t_{6k+8} &= t_{6k+5} + \frac{1}{3^{6k+6}} + \frac{0}{3^{6k+7}} + \frac{0}{3^{6k+8}}\\ &= t_{6k+5} + \frac{0}{3^{6k+6}} + \frac{0}{3^{6k+7}} + \frac{9}{3^{6k+8}}\\ &= t'_{6k+5} + \frac{1}{3^{6k+6}} + \frac{9}{3^{6k+7}} + \frac{0}{3^{6k+8}}\\ &= t'_{6k+5} + \frac{0}{3^{6k+6}} + \frac{9}{3^{6k+7}} + \frac{9}{3^{6k+8}}. \end{aligned} \tag{11}$$

Then $t_{6k+5} - t'_{6k+5} = \frac{1}{3^{6k+5}}$, so $t_{6k+5} = t_{6k+4} = \frac{1}{3^{6k+5}}$ is the representation of t_{6k+5}.

If $t'_{6k+5} = t_{6k+4} + \frac{0}{3^{6k+5}}$ is the representation of t'_{6k+5}, then

$$\#\langle t(6k+8)\rangle = 4\#\langle t(6k+4)\rangle \leqslant 4F_{6k+4} = F_{6k+8}.$$

If $t'_{6k+5} = t'_{6k+4} + \frac{9}{3^{6k+5}}$ is the other representation of t'_{6k+5}. Then $t_{6k+4} - t'_{6k+4} = \frac{3}{3^{6k+4}}$. By Lemma 2.3 we have two cases.

Case 1. If both of t_{6k+4}, t'_{6k+4} have the unique representation through point t_{6k+4} in supp μ_{6k+3}. Then

$$\#\langle t(6k+8)\rangle \leqslant 6\#\langle t(6k+3)\rangle \leqslant 6F_{6k+3} = F_{6k+8}.$$

Case 2. If both of t_{6k+4}, t'_{6k+4} have two representations through points in supp μ_{6k+3},

$$\begin{aligned} t_{6k+4} &= t_{6k+3} + \frac{0}{3^{6k+4}} = t''_{6k+3} + \frac{9}{3^{6k+4}},\\ t'_{6k+4} &= t'_{6k+3} + \frac{0}{3^{6k+4}} = t'''_{6k+3} + \frac{9}{3^{6k+4}}. \end{aligned} \tag{12}$$

Then $t'_{6k+3} - t'''_{6k+3} = \frac{3}{3^{6k+3}}$. By Lemma 2.3 we have two cases.

a) If t'_{6k+3} and t'''_{6k+3} have the unique representation through points t_{6k+2} and t'_{6k+2} in supp μ_{6k+2}, respectively. It implies $t_{6k+2} - t'_{6k+2} = \frac{1}{3^{6k+2}}$. Hence $t_{6k+2} = t_{6k+1} + \frac{1}{3^{6k+2}}$.

If $t'_{6k+2} = t_{6k+1} + \frac{0}{3^{6k+2}}$ is the unique representation, then

$$\#\langle t(6k+8)\rangle = 12\#\langle t(6k+1)\rangle \leqslant 12F_{6k+1} < F_{6k+8}.$$

If t'_{6k+2} has two representations, then $\#\langle t'(6k+2)\rangle \geqslant 2\#\langle t(6k+2)$. Thus,

$$\begin{aligned}\#\langle t(6k+8)\rangle &= 6\#\langle t(6k+2)\rangle + 6\#\langle t'(6k+2)\rangle \\ &\leqslant (3+6)\#\langle t'(6k+2) \leqslant 9F_{6k+2} = F_{6k+8}.tag13\end{aligned}$$

b) If both of t'_{6k+3} and t'''_{6k+3} have two representations through points in supp μ_{6k+2}, then it implies $t_{6k+2} - t'_{6k+2} = t''_{6k+2} - t'''_{6k+2} = \frac{1}{3^{6k+2}}$. Hence $\#\langle t'_{6k+4}\rangle \geqslant 2\#\langle t_{6k+4}\rangle$.

Then we have

$$\begin{aligned}F_{6k+5} \geqslant \#\langle t'_{6k+5}\rangle &= \#\langle t'_{6k+4}\rangle + \#\langle t_{6k+4}\rangle \\ &\geqslant 2\#\langle t_{6k+4}\rangle + \#\langle t_{6k+4}\rangle = 3\#\langle t_{6k+4}\rangle. \end{aligned} \tag{14}$$

It implies $\#\langle t_{6k+4}\rangle \leqslant \frac{1}{3}F_{6k+5}$. Therefore

$$\begin{aligned}\#\langle t(6k+8)\rangle &= 2\#\langle t'(6k+5)\rangle + 2\#\langle t(6k+5)\rangle \\ &= 2\#\langle t'(6k+5)\rangle + 2\#\langle t(6k+4)\rangle \\ &\leqslant (2+\frac{1}{3})F_{6k+5} = F_{6k+8}. \end{aligned} \tag{15}$$

2. If $y_{6k+4} = 1$, by similar argument as the proof of the cases $i \neq 2$, we have

$$\#\langle t(6k+8)\rangle \leqslant F_{6k+8}.$$

The proposition is proved.

4. Proof of the main theorem

4.1. Claim. For $s \in$ supp μ is defined by $s = S(x^0)$, where x^0 is in Proposition 3.6, We have $\alpha(s) = \frac{2}{3}$.

Proof. For any $n \geqslant 6$, there is $k \in N$ such that $6k \leqslant n \leqslant 6(k+1)$. By Proposition 3.7, we have

$$9^k \leqslant \#\langle s_n\rangle \leqslant 9^{k+1}.$$

It implies

$$\frac{|\log 9^k\ 3^{-6(k+1)}|}{6k\log 3} \geqslant \frac{|\log \mu_n(s_n)|}{n\log 3} \geqslant \frac{|\log 9^{k+1}\ 3^{-6k}|}{6(k+1)\log 3}.$$

Passing to the limit we get

$$\alpha(s) = \lim_{n\to\infty} \frac{|\log \mu_n(s_n)|}{n \log 3} = \frac{2}{3}.$$

The claim is proved.

4.2. Claim. $\overline{\alpha} = 1$, $\underline{\alpha} = \frac{2}{3}$.

Proof. For any prime sequence $x = (x_1, x_2, \dots)$, for examples $x = (x_1, x_2, \dots) = (0, 0, \dots)$, we have $\#\langle s_n \rangle = 1$ for every n, where $s_n = \sum_{i=1}^{n} 3^{-i} x_i$. Therefore, by Proposition 2.8 we get

$$\overline{\alpha} = \overline{\alpha}(s) = \lim_{n\to\infty} \frac{|\log \mu_n(s_n)|}{n \log 3} = 1,$$

where $s = S(x)$.

From Claim 4.1 we have

$$\underline{\alpha} \leqslant \frac{2}{3}.$$

For any $n \in N$, $n = 6k + i, k \in N, i = 0, 1, 2, 3, 4, 5$, we have

$$\#\langle t_n \rangle = \#\langle t_{6k+i} \rangle \leqslant 54 F_{6k} = 6.9^k.$$

Hence, $\mu_n(t_n) = \mu_{6k+i}(t_{6k+i}) \leqslant 3^{-(6k+i)} 6.9^k = 2.3^{-4k+(1-i)}$. We have

$$\lim_{n\to\infty} \frac{|\log \mu_{6k+i}(t_{6k+i})|}{(6k+i)\log 3} \geqslant \lim_{n\to\infty} \frac{|\log 2.3^{-4k+(1-i)}|}{(6k+i)\log 3} = \frac{2}{3} \tag{16}$$

for all $i = 0, 1, 2, 34, 5$, where t_n be n - partial sum of t. So we get

$$\underline{\alpha} \geqslant \frac{2}{3}.$$

Therefore

$$\underline{\alpha} = \frac{2}{3}.$$

The claim is proved.

To complete the proof of our Main Theorem it remains to prove the following claim.

4.3. Claim. For any $\beta \in (\frac{2}{3}, 1)$ there exists $s \in \operatorname{supp} \mu$ for which $\alpha(s) = \beta$.

Proof. Since $\beta \in (\frac{2}{3}, 1)$, there is $r \in (0, 1)$ such that $\beta = \frac{2}{3} r + (1 - r)1 = 1 - \frac{r}{3}$. For $i = 1, 2, \dots$, define

$$k_i = \begin{cases} 6i & \text{if } i \text{ is odd;} \\ [\frac{6i(1-r)}{r}] & \text{if } i \text{ is even.} \end{cases}$$

Let $n_j = \sum_{i=1}^{j} k_i$ and let

$$E_j = \{i : i \leqslant j \text{ and i is even}\}\ ;\ O_j = \{i : i \leqslant j \text{ and i is odd}\},$$

$$e_j = \sum_{i \in E_j} k_i \ ; \ o_j = \sum_{i \in O_j} k_i.$$

Then

$$n_j = o_j + e_j.$$

Similar proof as the proof of Claim 3.2 in [11], we get

$$\lim_{j \to \infty} \frac{j}{n_j} = 0 \ ; \ \lim_{j \to \infty} \frac{n_{j-1}}{n_j} = 1 \text{ and } \lim_{j \to \infty} \frac{o_j}{n_j} = r.$$

We define $s \in \operatorname{supp} \mu$ by $s = S(x)$, where

$$x = (\underbrace{1,1,1,1,0,0}_{k_1=6}\underbrace{0,0,\ldots,0}_{k_2}\underbrace{1,1,1,1,0,0,1,1,1,1,0,0,1,1,1,1,0,0}_{k_3=18}\underbrace{0,0,\ldots,0}_{k_4},\ldots). \quad (17)$$

Note that, for $i \in O_j$, we have $\#\langle s_{k_i} \rangle = F_{6i} = 9^i$. For $s \in \operatorname{supp} \mu$ is defined by x in (16) and for $n_{j-1} \leqslant n < n_j$, by the multiplication principle, we have

$$\prod_{i \in O_{j-1}} \#\langle s_{k_i} \rangle \leqslant \#\langle s_n \rangle \leqslant \prod_{i \in O_j} \#\langle s_{k_i} \rangle.$$

Hence, by Claim 3.4 yield

$$9^{\frac{O_{j-1}}{6}} \leqslant \#\langle s_n \rangle \leqslant 9^{\frac{O_j}{6}},$$

which implies

$$\frac{\log 9^{\frac{O_{j-1}}{6}}}{n_j \log 3} \leqslant \frac{\log \#\langle s_n \rangle}{n \log 3} \leqslant \frac{\log 9^{\frac{O_j}{6}}}{n_{j-1} \log 3}.$$

Hence

$$\lim_{n \to \infty} \frac{\log \#\langle s_n \rangle}{n \log 3} = \frac{r}{3}.$$

Therefore

$$\alpha(s) = \lim_{n \to \infty} \frac{|\log \#\langle s_n \rangle 3^{-n}|}{n \log 3} = 1 - \lim_{n \to \infty} \frac{\log \#\langle s_n \rangle}{n \log 3} = 1 - \frac{r}{3} = \beta.$$

The claim is proved.

From Claim 4.2 and 4.3 the Main Theorem follows

References

1. Truong Thuy Duong and Vu Hong Thanh, Singularity of Fractal Measure associated with the (0,1,7)-Problem. *Vietnam National university, Hanoi*, $\mathbf{N}_0\mathbf{2}$(2005), 7-19.
2. K. J. Falconer, *Techniques in Fractal Geometry,* John Wiley & Sons, 1997.
3. K. J. Falconer, *Fractal Geometry, Mathematical Foundations and Applications,*John Wiley & Sons, 1993.

4. A. Fan, K. S. Lau and S. M. Ngai, Iterated function systems with overlaps, *Asian J. Math.* **4**(2000), 527 - 552.
5. T. Hu, The local dimensions of the Bernoulli convolution associated with the golden number, *Trans. Amer. Math. Soc.* **349**(1997), 2917 - 2940.
6. T. Hu and N. Nguyen, Local dimensions of fractal probability measures associated with equal probability weight, *Preprint.*
7. T. Hu, N. Nguyen and T. Wang, Local dimensions of the probability measure associated with the (0, 1, 3) - problem, *Preprint.*
8. T. Hu, Some open questions related to Probability, Fractal, Wavelets, *East - West J. of Math.* Vol 2, $\mathbf{N}_0\mathbf{1}$(2000), 55-71.
9. J. C. Lagarias and Y. Wang, Tiling the line with translates of one tile, *Inventions Math.* **124**(1996), 341 - 365.
10. S. M. Ngai and Y. Wang, Hausdorff dimention of the self - similar sets with overlaps,*J. London Math. Soc.* (to appear).
11. Le Xuan Son, Pham Quang Trinh and Vu Hong Thanh. Local Dimension of Fractal Measure associated with the (0,1,a)-Problem, the case $a = 6$. *Vietnam National university, Hanoi*, $\mathbf{N}_1$(2005), 31-44.