STABILITY OF THE ELASTOPLASTIC THIN ROUND CYLINDRICAL SHELLS SUBJECTED TO TORSIONAL MOMENT AT TWO EXTREMITIES

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Abstract. An elastic stability problem of the thin round cylindrical shells subjected to torsional moment at two extremities has been investigated in the paper [6]. By the small elastoplastic deformation theory and by the flow theory, this problem again has been studied in [2] and [4]. Basing on the theory of elastoplastic processes the above mentioned problem has been solved by approach simulation of instability form of the cylinder (see [1],[5]). In this paper, the solution of problem in the real bending form of structure has been found. We have also established the relations for determining critical force. Some numerical results for a linear hardening material have been given and discussed.

1. Stability problem of cylindrical shell

Let us consider a thin round cylindrical shell of strength L, thickness h and radius of the middle surface equal to R. We choose a orthogonal coordinate system $Oxyz$ so that axis x lies along the generatrix of cylindrical shell while $y = R\theta$ with θ- the angle circular arc and z in direction of the normal to cylindrical shell.

Suppose that cylindrical shell has the simply supported boundary constraints at $x = 0, x = L$ and subjected to torsion by a couple of moments $M_k = 2\pi R^2 hp, p = p(t)$ with t-loading parameter. Moreover, we assume that material is incompressible and don't take into account the unloading in the cylindrical shell. We have to find the critical values $t = t_*$ and $p_* = p(t_*)$ which at that time an instability of the structure appears. We use the criterion of bifurcation of equilibrium states to study the proposed problem.

2. Fundamental equations of the stability problem

2.1. Pre-buckling process

At the any moment in the pre-buckling state, we have

$$\sigma_{12} = -p \ , \quad \sigma_{ij} = 0 \quad \forall i \neq 1 \ , \ j \neq 2.$$
$$\sigma_u = \sqrt{3}|\sigma_{12}| = \sqrt{3}p.$$

The components of the strain velocity tensor determined respectively

$$\dot{\varepsilon}_{12} = -\frac{3\dot{p}}{2\phi'} \ , \ \ \dot{\varepsilon}_{ij} = 0 \ \ \forall i \neq 1, j \neq 2 \ \ , \ \ \phi' \equiv \phi'(s).$$

The arc-length of the strain trajectory is calculated by the formula

$$\frac{ds}{dt} = \frac{2}{\sqrt{3}}|\dot{\varepsilon}_{12}| = \frac{\sqrt{3}\dot{p}}{\phi'}.$$

It is seen from here that $\phi(s) = \sqrt{3}p$ or $s = \phi^{-1}(\sqrt{3}p)$.

2.2. Post-buckling process and boundary conditions

The system of stability equations of the thin cylindrical shell established in [1,5] are written in form

$$\alpha_1\frac{\partial^4\delta w}{\partial x^4} + \alpha_3\frac{\partial^4\delta w}{\partial x^2\partial y^2} + \alpha_5\frac{\partial^4\delta w}{\partial y^4} - \frac{9}{h^2N}\left(-2p\frac{\partial^2\delta w}{\partial x\partial y} + \frac{1}{R}\frac{\partial^2\varphi}{\partial x^2}\right) = 0; \qquad (2.1)$$

$$\beta_1\frac{\partial^4\varphi}{\partial x^4} + \beta_3\frac{\partial^4\varphi}{\partial x^2\partial y^2} + \beta_5\frac{\partial^4\varphi}{\partial y^4} + \frac{N}{R}\frac{\partial^2\delta w}{\partial x^2} = 0, \qquad (2.2)$$

where

$$\alpha_1 = 1,\ \alpha_3 = 1 + \frac{\phi'}{N},\ \alpha_5 = 1;$$

$$\beta_1 = 1,\ \beta_3 = 3\frac{N}{\phi'} - 1,\ \beta_5 = 1;$$

$$\phi' = \phi'(s),\ N = \frac{\sigma_u}{s}.$$

The simply supported boundary conditions give us

$$\delta w\Big|_{x=0,x=L} = 0,\ \frac{\partial^2\delta w}{\partial x^2}\Big|_{x=0,x=L} = 0. \qquad (2.3)$$

3. Solving method

From the experimental results (see [2]) and the similar form of solution in [3], we find the real deflection δw in form

$$\delta w = A_1\cos\frac{\pi x}{L}\cos\frac{n}{R}(y + \gamma x), \qquad (3.1)$$

where γ is the tangent of skew angle of summit of waves in comparison with the generatrix of cylindrical shell, n-number of waves in direction of round arc. The just chosen solution satisfies the simply supported boundary condition in the sense of Saint-Venant at $x = 0,\ x = L$. In fact,

$$\int_0^{2\pi R}\delta w(0,y)dy = \int_0^{2\pi R}A_1\cos\frac{ny}{R}dy = 0$$

$$\int_0^{2\pi R}\delta w(L,y)dy = -\int_0^{2\pi R}A_1\cos\frac{n}{R}(y+\gamma L)dy = 0$$

$$\int_0^{2\pi R}\frac{\partial^2\delta w}{\partial x^2}(0,y)dy = -\int_0^{2\pi R}A_1\left[\left(\frac{\pi}{L}\right)^2 + \left(\frac{n\gamma}{R}\right)^2\right]\cos\frac{ny}{R}dy = 0$$

$$\int_0^{2\pi R}\frac{\partial^2\delta w}{\partial x^2}(L,y)dy = \int_0^{2\pi R}A_1\left[\left(\frac{\pi}{L}\right)^2 + \left(\frac{n\gamma}{R}\right)^2\right]\cos\frac{n}{R}(y+\gamma L)dy = 0$$

In order to solve advantageously the problem, we rewrite the expression of δw in form

$$\delta w = \frac{A_1}{2}\cos\left(\frac{ny}{R} + mx\right) + \frac{A_1}{2}\cos\left(\frac{ny}{R} + jx\right), \tag{3.2}$$

where

$$m = \frac{n\gamma}{R} + \frac{\pi}{L} \ \ , \ j = \frac{n\gamma}{R} - \frac{\pi}{L}.$$

Now we find the particular solution φ of equation (2.2) in form

$$\varphi = B_1\cos\left(\frac{ny}{R} + mx\right) + B_2\cos\left(\frac{ny}{R} + jx\right). \tag{3.3}$$

Substituting (3.2), (3.3) into (2.2) and comparing the coefficients of $cos\left(\frac{ny}{R} + mx\right)$ and $\cos\left(\frac{ny}{R} + jx\right)$, we obtain $B_1 = A_1 B_{01},\ B_2 = A_1 B_{02}$ where

$$B_{01} = \frac{Nm^2}{2R}\frac{1}{\beta_1 m^4 + \beta_3 m^2\left(\frac{n}{R}\right)^2 + \beta_5\left(\frac{n}{R}\right)^4};$$
$$B_{02} = \frac{Nj^2}{2R}\frac{1}{\beta_1 j^4 + \beta_3 j^2\left(\frac{n}{R}\right)^2 + \beta_5\left(\frac{n}{R}\right)^4}.$$

Putting δw and φ into (2.1) and because of the condition on the existence of non-trivial solution, we get

$$\frac{9np}{h^2NR} = \frac{\alpha_1 m^3}{2} + \frac{\alpha_3 m}{2}\left(\frac{n}{R}\right)^2 + \frac{\alpha_5}{2m}\left(\frac{n}{R}\right)^4 + \frac{9mB_{01}}{h^2NR}; \tag{3.4}$$
$$\frac{9np}{h^2NR} = \frac{\alpha_1 j^3}{2} + \frac{\alpha_3 j}{2}\left(\frac{n}{R}\right)^2 + \frac{\alpha_5}{2j}\left(\frac{n}{R}\right)^4 + \frac{9jB_{02}}{h^2NR}. \tag{3.5}$$

We receive from here the expression for determining critical load

$$\frac{\alpha_1 m^3}{2} + \frac{\alpha_3 m}{2}\left(\frac{n}{R}\right)^2 + \frac{\alpha_5}{2m}\left(\frac{n}{R}\right)^4 + \frac{9mB_{01}}{h^2NR} = \frac{\alpha_1 j^3}{2} + \frac{\alpha_3 j}{2}\left(\frac{n}{R}\right)^2 + \frac{\alpha_5}{2j}\left(\frac{n}{R}\right)^4 + \frac{9jB_{02}}{h^2NR}.$$

Substituting the expression of B_{01} and B_{02} into the just obtained equation, we have

$$\frac{\alpha_1 m^3}{2} + \frac{\alpha_3 m}{2}\left(\frac{n}{R}\right)^2 + \frac{\alpha_5}{2m}\left(\frac{n}{R}\right)^4 + \frac{9m^3}{2R^2h^2}\frac{1}{\beta_1 m^4 + \beta_3 m^2\left(\frac{n}{R}\right)^2 + \beta_5\left(\frac{n}{R}\right)^4} =$$
$$= \frac{\alpha_1 j^3}{2} + \frac{\alpha_3 j}{2}\left(\frac{n}{R}\right)^2 + \frac{\alpha_5}{2j}\left(\frac{n}{R}\right)^4 + \frac{9j^3}{2R^2h^2}\frac{1}{\beta_1 j^4 + \beta_3 j^2\left(\frac{n}{R}\right)^2 + \beta_5\left(\frac{n}{R}\right)^4} \tag{3.6}$$

Remarks

a) If material is elastic, i.e. $N = 3G,\ \phi' = 3G$, we get

$$\alpha_1 = \alpha_5 = 1,\ \ \alpha_3 = 2;\ \ \beta_1 = \beta_5 = 1,\ \ \beta_3 = 2.$$

The expression (3.4) and (3.5) are of the form

$$\frac{2p}{3G} = \frac{(m^2R^2+n^2)^2}{9mnR}\left(\frac{h}{R}\right)^2 + \frac{m^3R^3}{n(m^2R^2+n^2)^2}\ ,$$
$$\frac{2p}{3G} = \frac{(j^2R^2+n^2)^2}{9jnR}\left(\frac{h}{R}\right)^2 + \frac{j^3R^3}{n(j^2R^2+n^2)^2}\ .$$

These results coincides with the previous well-known ones (see [2]).

b) If material is small elasto-plastic i.e. $\phi' = E_t,\quad N = E_c$, then

$$\alpha_1 = \alpha_5 = 1,\quad \alpha_3 = 1 + \frac{E_t}{E_c};\quad \beta_1 = \beta_5 = 1,\quad \beta_3 = 3\frac{E_c}{E_t} - 1\ .$$

The expression (3.4) and (3.5) return to the results presented in [2].

4. Linear hardening material

In this case, we have

$$\phi' \equiv g = const,\quad \sigma_u = 3Gs_0 + (s - s_0)\phi' = gs + (3G-g)s_0\ \ .$$

Putting $\lambda = (3G-g)s_0$, we obtain $\sigma_u = gs + \lambda$,

$$\alpha_1 = \alpha_5 = 1,\quad \alpha_3 = 1 + \frac{\phi'}{N} = 1 + \frac{gs}{\sigma_u} = \frac{2gs+\lambda}{gs+\lambda}$$
$$\beta_1 = \beta_5 = 1,\quad \beta_3 = 3\frac{N}{\phi'} - 1 = 3\frac{\sigma_u}{gs} - 1 = 2 + \frac{3\lambda}{gs}.$$

Substituting these values into (3.6), we obtain equation

$$a_1 + \frac{a_2 s}{gs+\lambda} + \frac{a_3 gs}{a_4 gs + a_5} = b_1 + \frac{b_2 s}{gs+\lambda} + \frac{b_3 gs}{b_4 gs + b_5}, \tag{4.1}$$

where

$$a_1 = \frac{1}{2}m^3 + \frac{1}{2}m\left(\frac{n}{R}\right)^2 + \frac{1}{2m}\left(\frac{n}{R}\right)^4\ ,\quad a_5 = 3\lambda m^2\left(\frac{n}{R}\right)^2,$$
$$a_2 = \frac{1}{2}mg\left(\frac{n}{R}\right)^2\ ,\quad a_3 = \frac{9m^3}{2R^2h^2}\ ,\quad a_4 = m^4 + 2m^2\left(\frac{n}{R}\right)^2 + \left(\frac{n}{R}\right)^4,$$
$$b_1 = \frac{1}{2}j^3 + \frac{1}{2}j\left(\frac{n}{R}\right)^2 + \frac{1}{2j}\left(\frac{n}{R}\right)^4\ ,\quad b_5 = 3\lambda j^2\left(\frac{n}{R}\right)^2,$$
$$b_2 = \frac{1}{2}jg\left(\frac{n}{R}\right)^2\ ,\quad b_3 = \frac{9j^3}{2R^2h^2}\ ,\quad b_4 = j^4 + 2j^2\left(\frac{n}{R}\right)^2 + \left(\frac{n}{R}\right)^4.$$

Transforming (4.1), we receive three-order algebraic equation of s

$$As^3 + Bs^2 + Cs + D = 0, \tag{4.2}$$

where

$$A = b_4g^2(a_1a_4g + a_3g + a_2a_4) - a_4g^2(b_1b_4g + b_3g + b_2b_4)$$

$$B = b_5g(a_1a_4g+a_3g+a_2a_4)-a_5g(b_1b_4g+b_3g+b_2b_4)+b_4g(\lambda a_1a_4g+\lambda a_3g+a_1a_5g+a_2a_5)-$$

$$-a_4g(\lambda b_1b_4g + \lambda b_3g + b_1b_5g + b_2b_5)$$

$$C = b_5(\lambda a_1a_4g+\lambda a_3g+a_1a_5g+a_2a_5)-a_5(\lambda b_1b_4g+\lambda b_3g+b_1b_5g+b_2b_5)+\lambda g(b_4a_1a_5-a_4b_1b_5)$$

$$D = \lambda(a_1a_5b_5 - b_1b_5a_5) \ .$$

It is seen that for each given material and each determined value of γ , with n changing from 1 to k , then we can solve equation (4.2) for finding s_n . After that we choose value $s_{min} = min(s_1, s_2, ..., s_k)$. Finally, the critical load is found by putting s_{min} into the expression of σ_u

$$\sigma_{umin} = \phi' s_{min} + (3G - \phi')s_0 \ , \quad p_{min} = \frac{1}{\sqrt{3}}\sigma_{umin} \ .$$

R/h	$s.10^3$	σ_u^*(Mpa)
25	18.4	750.61
28	11.68	610.83
31	8.3	540.53
34	6.3	498.93
37	4.978	471.43
40	4.061	452.36
43	3.4	438.61
46	2.879	427.77
49	2.48	419.47
52	2.164	413
55	1.907	407.55
58	1.694	403.12
61	1.516	399.42
64	1.366	396.3
67	1.238	393.64
70	1.128	391.35
73	1.032	389.35
76	0.948	387.61
79	0.8745	386.08

Table 1. The results basing on the elasto-plastic theory

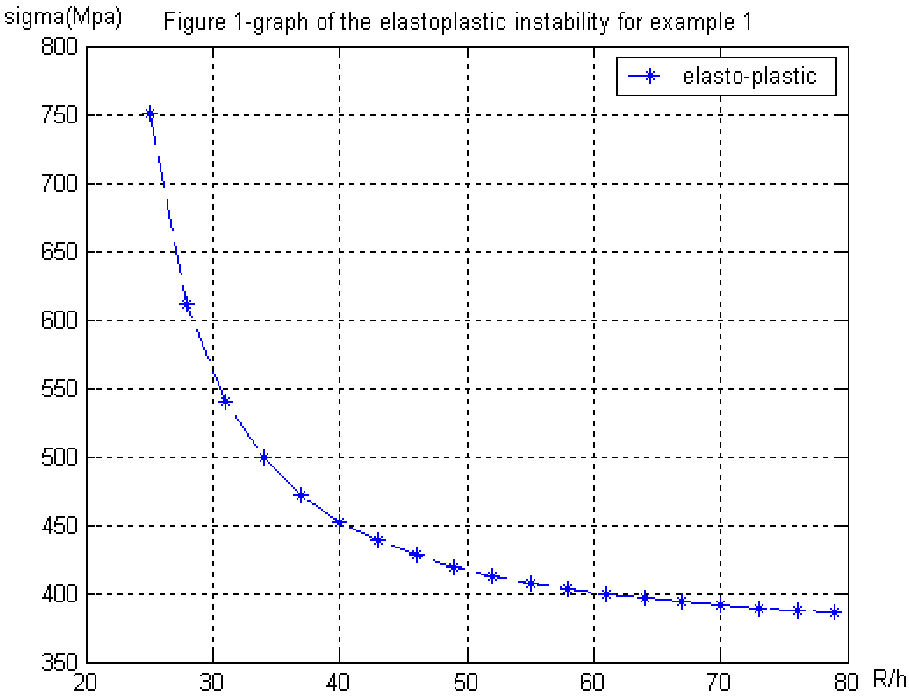
5. Numerical results and discussion

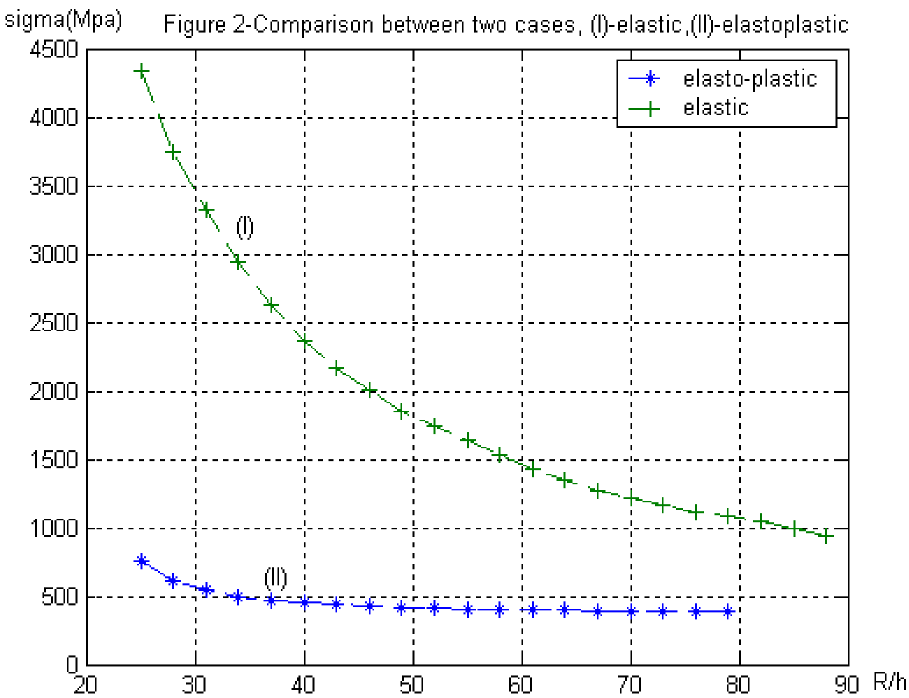
Example 1: Let us consider material with the characteristics as follow $3G = 2,6.10^5 Mpa$; $\phi' = 0,208.10^5 Mpa$; $\gamma = 0,5$; $R = 4$; $L = 10$; n from 1 to 14 (L, R, h in metres). The numerical results basing on the theory of elasto-plastic processes are given in table 1.

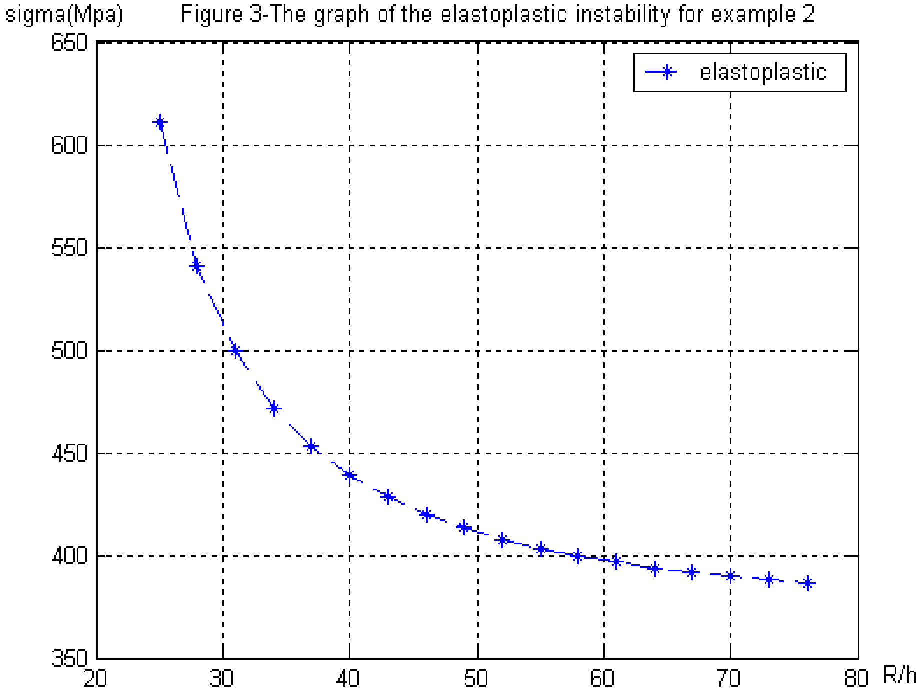
The calculating results based on the elastic theory are presented in table 2. Figure 1 is graph of the elasto-plastic instability case . The comparison between the elasto-plastic instability and the elastic instability is introduced in figure 2.

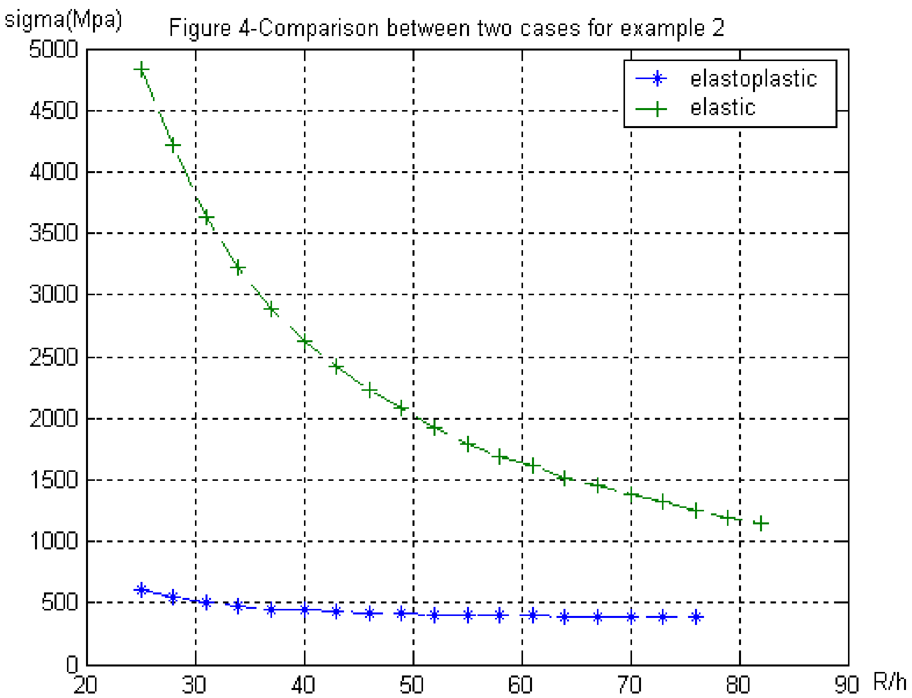
R/h	$s.10^3$	σ_u^*(Mpa)
25	16.7	4342
28	14.4	3744
31	12.8	3328
34	11.3	2938
37	10.1	2626
40	9.1	2366
43	8.3	2158
46	7.7	2002
49	7.1	1846
52	6.7	1742
55	6.3	1638
58	5.9	1534
61	5.5	1430
64	5.2	1352
67	4.9	1274
70	4.7	1222
73	4.5	1170
76	4.3	1118
79	4.2	1092
82	4	1042
85	3.8	988
88	3.6	936

Table 2. The calculating results according to the elastic theory


sigma(Mpa)
Figure 1-graph of the elastoplastic instability for example 1
elasto-plastic
800
750
700
650
600
550
500
450
400
350
20
30
40
50
60
70
80
R/h


sigma(Mpa)
Figure 2-Comparison between two cases, (I)-elastic,(II)-elastoplastic
elasto-plastic
elastic
(I)
(II)
4500
4000
3500
3000
2500
2000
1500
1000
500
0
20
30
40
50
60
70
80
90
R/h


sigma(Mpa)
Figure 3-The graph of the elastoplastic instability for example 2
elastoplastic
650
600
550
500
450
400
350
20
30
40
50
60
70
80
R/h


sigma(Mpa)
Figure 4-Comparison between two cases for example 2
elastoplastic
elastic
5000
4500
4000
3500
3000
2500
2000
1500
1000
500
0
20
30
40
50
60
70
80
90
R/h

Example 2: Let us consider material with the characteristics as follows

$$3G = 2,6.10^5 Mpa \ , \quad \phi' = 0,208.10^5 Mpa \ , \quad \gamma = \frac{\sqrt{3}}{3} \ , \quad R = 5 \ , \quad L = 10$$

The results of calculation are sketched by graphs in figures 3 and 4.

Discussion

The above received results lead us to some remarks as follows

a) The critical loads determined according to the elastic theory are much greater than those according to the theory of elasto-plastic processes when the thickness of cylindrical shell is greater. Because these don't exactly describe mechanical characteristics, investigating must be based on the theory of elasto-plastic processes for thicker cylindrical shells.

b) When the slenderness of cylindrical shell reachs a determined value, the difference between the critical loads found by basing on two theories is very little. Therefore for the slender cylindrical shells, calculating based on the elastic theory is reliable.

c) The expression of deflection δw in (3.1) has exactly described real bending form of structure.

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References

1. Dao Huy Bich, *Theory of elastoplastic processes*, Vietnam National University Publishing House, Hanoi 1999 (in Vietnamese).
2. Volmir A.S.*Stability of deformable systems* , Moscow 1963 (in Russian).
3. Dao Huy Bich, On the elastoplastic stability of a plate under shear forces, taking into account its real bending form, *Vietnam Journal of Mech, NCST of Vietnam,* Vol 23, **No 1**(2001), pp 6-16.
4. G.Gerard, Plastic stability theory of thin shell, *J.Aeron. Sci.,* 24, **No 4**(1957), pp 264-279.
5. Dao Van Dung, Stability problem outside elastic limit according to the theory of elastoplastic deformation processes, *Ph. D. Thesis,* Hanoi 1993 (in Vietnamese).
6. L.H. Donnell, Stability of thin walled tubes under torsion, *NACA Rept.* **No 479**(1933).