POLYNOMIAL APPROXIMATION ON POLYDISKS

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Abstract. In this paper we give results about polynomial approximation on the closed polydisk in $\mathbb{C}^n$.

1. Introduction

Let X be a compact subset of $\mathbb{C}^n$. By $C(X)$ we denote the space of all continuous complex-valued functions on X, with norm $\|f\|_X = \max\{|f(z)| : z \in X\}$, and let $P(X)$ denote the closure of set of polynomials in $C(X)$. The polynomially convex hull of X will be denoted by $\hat{X}$ and difined by

$$\hat{X} = \{z \in \mathbb{C}^n : |p(z)| \leqslant \|p\|_X \quad \text{for every polynomial } p\}.$$

X is called polynomially convex if $X = \hat{X}$. One necessary condition for $P(X) = C(X)$ is X is polynomially convex.

Let M be real manifold in $\mathbb{C}^n$. We say that M has *totally* real if M has no complex tangent vectors, i.e. $(T_aM) \cap i(T_aM) = \{0\}$. It is well-known that every continuous function on compact subset X of totally real manifold M is the uniform limit of sequence of polynomials, i.e. $P(X) = C(X)$.

Let D be small closed polydisk in $\mathbb{C}^n$, centered at the origin and $f_1, f_2, ..., f_m \in C(D)$ By $[z_1, z_2, ..., z_n, f_1, f_2, ..., f_m; D]$ we denote the function algebra consisting of uniform limits on D of all polynomials in $z_1, z_2, ..., z_n$ and $f_1, f_2, ..., f_m$. The problem is that to find conditions of $f_1, ..., f_m$ such that $[z_1, z_2, ..., z_n, f_1, f_2, ..., f_m; D] = C(D)$. J.Wermer, Nguyen Quang Dieu, P.J. de Paepe,... have many results if D is disk. Nguyen Quang Dieu and P.J. de Paepe have shown that $[z, f(z); D]$ $=C(D)$ for some choices of f, while for other choices of f to have $[z, f(z); D] \neq C(D)$ (see [4], [5], [6], [7]). In the general, H. Alexander and J.Wermer [1] (Theorem 17.5) used the result about approximation of totally real manifold to proved the following results.

Theorem 1.1. *Suppose $R_1(z), ..., R_n(z)$ are complex-value continuoustly differentiable function on D satisfy conditions*

$$|R(z) - R(z')| \leqslant k|z - z'| \quad \forall z, z' \in D,$$

where $0 \leqslant k < 1$, $R(z) = (R_1(z), ..., R_n(z))$ and $|w| = \sqrt{|w_1|^2 + ... + |w_n|^2}$ with $w = (w_1, ..., w_n) \in \mathbb{C}^n$. Then

$$[z_1, ..., z_n, \overline{z}_1 + R_1(z), ..., \overline{z}_n + R_n(z); D] = C(D).$$

The line of proof of Wermer is to prove the graph

$$X = \{(z_1, ..., z_n, \overline{z}_1 + R_1(z), ..., \overline{z}_n + R_n(z)) : z = (z_1, ..., z_n) \in D\}$$

is totally real manifold. In this paper we give the generalize result of Wermer. But the line of proof is not the same of Wermer's, because most of our graph is not totally real at the origin. Our proof use the theorem of Stout [3]:

Theorem 1.2. *Suppose that:*

(1) X_1 and X_2 are compact subsets of $\mathbb{C}^n$ with $P(X_1) = C(X_1)$ and $P(X_2) = C(X_2)$;

(2) Y_1 and Y_2 are polynomially convex subsets of $\mathbb{C}$ such that 0 is boundary point of both Y_1 and Y_2 , and $Y_1 \cap Y_2 = 0$;

(3) p is a polynomial such that $p(X_1) \subset Y_1$ and $p(X_2) \subset Y_2$;

(4) $p^{-1}(0) \cap (X_1 \cup X_2) = X_1 \cap X_2$. Then $P(X_1 \cup X_2) = C(X_1 \cup X_2)$.

For an extensive survey on Stout's theorem and its proof we refer the reader to [7] and [10], for basis materrial on polynomial convexity and totally real manifold the readers may consult [1] and [2].

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2. The main results

We now come to the main results of the paper.

Theorem 2.1. *Let $k_1, k_2, ..., k_{2n}$ be positive integers and $\gcd(k_i, k_j) = 1$ (k_i, k_j are co-prime), $\forall i \neq j$ and D is sufficiently small polydisk centered at the origin in $\mathbb{C}^n$. Let $R : \mathbb{C}^n \to \mathbb{C}^n$ be a function satisfying conditions*

1) $R(z) = (R_1(z), ..., R_n(z))$ with $R_i(z)$ is continuoustly differentiable functions on D;

2) $R(z) = o(|z|)$;

3) $|R(z) - R(z')| \leqslant M|z - z'|, \forall z, z' \in D$ where $0 \leqslant M < 1$.

Then

$$[z_1^{k_1}, ..., z_n^{k_n}, (\overline{z_1} + R_1)^{k_{n+1}}, ..., (\overline{z_n} + R_n)^{k_{2n}}; D] = C(D).$$

Remark. The graph

$$X = \{(z_1^{k_1}, ..., z_n^{k_n}, (\overline{z_1} + R_1(z))^{k_{n+1}}, ..., (\overline{z_n} + R_n(z)^{k_{2n}}) : z = (z_1, ..., , z_n) \in D\}$$

is not totally real at the origin if there exists $k_{n+i} \neq 1$. (see [2], p. 2). By Theorem 2.1 it is polynomially convex if D small enough.

We need the following proposition.

Proposition 2.2. *Let X be a compact subset of C^m, and let $\pi : \mathbb{C}^m \to C^m$ be difined by*

$$\pi(z_1, z_2, ..., z_m) = (z_1^{k_1}, z_2^{k_2}, ..., z_m^{k_m}).$$

Let $\pi^{-1}(X) = X_{1,1,...,1} \cup ... \cup X_{i_1,i_2,...,i_m} \cup ... \cup X_{k_1,k_2,...,k_m}$, with $X_{1,1...,1}$ compact, and

$$X_{i_1,i_2,...,i_m} = \{(\rho_{k_1}^{i_1-1} z_1, \rho_{k_2}^{i_2-1} z_2, ..., \rho_{k_m}^{i_m-1} z_m) : (z_1, z_2, ..., z_m) \in X_{1,1,...,1}\}$$

for $1 \leqslant i_1 \leqslant k_1, ..., 1 \leqslant i_m \leqslant k_m$, where $\rho_{k_j} = \exp(\frac{2\pi i}{k_j})$ with $j = 1, ..., m$. If $P(\pi^{-1}(X)) = C(\pi^{-1}(X))$, then $P(X) = C(X)$.

Lemma 2.3. *Suppose $X_{i_1,i_2,...,i_m}$ are difined as in proposition 2.2 and let*

$$Q(z_1, z_2, ..., z_m) = \sum a_{j_1,...,j_m} z^{j_1} z^{j_2} ... z^{j_m}$$

by a polynomial in m variables. For each $X_{i_1,i_2,...,i_m}$ with $1 \leqslant i_1 \leqslant k_1$,..., $1 \leqslant i_m \leqslant k_m$ put

$$Q_{i_1,i_2,...,i_m}(z_1, ..., z_m) = Q(\rho_{k_1}^{i_1-1} z_1, ..., \rho_{k_m}^{i_m-1} z_m).$$

Then

$$\frac{1}{k_1 k_2 ... k_m} \sum Q_{i_1,...,i_m}(z_1, ..., z_m) = \sum a_{p_1 k_1,...,p_m k_m} z_1^{k_1 p_1} ... z_m^{k_m p_m}.$$

Proof. First, we assume that Q has the form $Q(z_1, ..., z_m) = a z_1^{s_1} ... z_m^{s_m}$. We have

$$\frac{1}{k_1 k_2 ... k_m} \sum Q_{i_1,...,i_m}(z_1, ..., z_m) = \frac{a}{k_1 ... k_m} z_1^{s_1} ... z_m^{s_m} \sum_{1 \leqslant i_1 \leqslant k_1, ..., 1 \leqslant i_m \leqslant k_m} \rho_{k_1}^{(i_1-1)s_1} ... \rho_{k_m}^{(i_m-1)s_m}$$

$$= \frac{a}{k_1 ... k_m} z_1^{s_1} ... z_m^{s_m} \prod_{j=1}^{m} \Big(\sum_{1 \leqslant i_j \leqslant k_j} (\rho_{k_j}^{s_j})^{i_j - 1} \Big).$$

If there exists $1 \leqslant j \leqslant m$ such that $s_j \neq p_j k_j$ then

$$\sum_{1 \leqslant i_j \leqslant k_j} (\rho_{k_j}^{s_j})^{i_j-1} = \frac{(\rho_{k_j}^{s_j})^{k_j} - 1}{\rho_{k_j}^{s_j} - 1} = \frac{(\rho_{k_j}^{k_j})^{s_j} - 1}{\rho_{k_j}^{s_j} - 1} = 0,$$

where $\rho_{k_j}^{k_j} = (\exp \frac{2\pi i}{k_j})^{k_j} - 1 = 0$. We obtain

$$\frac{1}{k_1 k_2 ... k_m} \sum Q_{i_1,...,i_n}(z_1, ..., z_m) = 0.$$

In the case $s_j = p_j k_j$, $\forall 1 \leqslant j \leqslant m$ we have

$$\sum_{1 \leqslant i_j \leqslant k_j} (\rho_{k_j}^{s_j})^{i_j-1} = \sum_{1 \leqslant i_j \leqslant k_j} (\rho_{k_j}^{k_j})^{p_j(i_j-1)} = k_j.$$

We conclude that

$$\frac{1}{k_1k_2...k_m}\sum Q_{i_1,...,i_m}(z_1,...,z_m)=\frac{a}{k_1...k_m}z_1^{s_1}...z_m{}^{s_m}\prod_{j=1}^{m}k_j=az_1^{p_1k_1}...z_m^{p_mk_m}.$$

Since every polynomial Q can write that the finitely sum of $az_1^{s_1}...z_m^{s_m}$, we have

$$\frac{1}{k_1k_2...k_m}\sum Q_{i_1,...,i_m}(z_1,...,z_m)=\sum a_{p_1k_1,...,p_mk_m}z_1^{k_1p_1}...z_m^{k_mp_m}.$$

Proof of proposition 2.2.

Let $f\in C(X)$. Then $f\circ\pi\in C(\pi^{-1}(X))$, so there is a polynomial Q in two variables, with $f\circ\pi\sim Q$ on $\pi^{-1}(X)$. In particular, this is true for $X_{i_1,i_2,...,i_m}$, so

$$f(z_1^{k_1},...,z_m^{k_m})\sim Q(\rho_{k_1}^{i_1-1}z_1,...,\rho_{k_m}^{i_m-1}z_m)=Q_{i_1,...,i_m}(z_1,...,z_m)\quad\text{on}\quad X_{1,...,1}.$$

It follows that

$$f(z_1^{k_1},...,z_m^{k_m})\sim\frac{1}{k_1...k_m}\sum Q_{i_1,...,i_m}(z_1,...,z_m)\quad\text{on}\quad X_{1,...,1}.$$

By Lemma 2.3 if

$$Q(z_1,...,z_m)=\sum a_{r_1,...,r_m}z_1^{r_1}...z_m^{r_m},$$

then the right hand side to equals $\sum a_{p_1k_1,...,p_mk_m}z_1^{k_1p_1}...z_m^{k_mp_m}$, so equals to $P(z_1^{k_1},...,z_m^{k_m})$, where P is a polynomial in m variables. We conclude that

$$f(z_1^{k_1},...z_m^{k_m})\sim P(z_1^{k_1},...z_m^{k_m})\quad\text{on}\quad X_{1,...,1}.$$

That is $f\sim P$ on X. So $P(X)=Q(X)$.

Proof of Theorem 2.1. First we show that the functions

$$z_1^{k_1},...,z_n^{k_n},(\overline{z_1}+R_1(z))^{k_{n+1}},...,(\overline{z_n}+R_n(z))^{k_{2n}}$$

separate points near the origin. Indeed, let $a=(a_1,...,a_n),b=(b_1,...b_n)\in D$ and $a\neq b$. Since $a\neq b$ there exist $1\leqslant i\leqslant n$ such that $a_i\neq b_i$. If $b_i\neq a_i\exp(\frac{2s\pi i}{k_i})$, $1\leqslant s\leqslant k_i-1$ then $z_i^{k_i}$ separates a_i,b_i.

Next if $b_i=a_i\exp(\frac{2s\pi i}{k_i})$ we show that $(\overline{z}_i+R_i(z))^{k+i}$ separates them. Now assume that $(\overline{z}+R_i(z))^{k_{n+i}}$ take the same value at a_i and b_i. We have

$$(\overline{a}_i+R_i(a))^{k_{n+i}}=(\overline{b}_i+R_i(b))^{k_{n+i}}.$$

We obtain

$$\overline{a}_i+R_i(a)=(\overline{a}_i\exp(\frac{2r\pi i}{k_i})+R_i(b))\exp(\frac{2t\pi i}{k_{n+i}})$$

for some $1 \leqslant r \leqslant k_i - 1$ and $1 \leqslant t \leqslant k_{n+i}$. It follows that

$$\overline{a}_i\Big(1 - \exp(\frac{2r\pi i}{k_i})\exp(\frac{2t\pi i}{k_{n+i}})\Big) = (-R_i(a) + R_i(b)\exp(\frac{2t\pi i}{k_{n+i}})).$$

Dividing both sides by $\overline{a}_i$ we obtain

$$\Big(1 - \exp(\frac{2r\pi i}{k_i})\exp(\frac{2t\pi i}{k_{n+i}})\Big) = \frac{-R_i(a) + R_i(b)\exp(\frac{2t\pi i}{k_{n+i}})}{\overline{a}_i.}$$

By the coprimeness of k_i and k_{n+i} we see that $\frac{r}{k_i} + \frac{t}{k_{n+i}}$ is not integer for all $1 \leqslant r \leqslant k_i - 1$ and $1 \leqslant t \leqslant k_{n+i}$, so $\Big(1 - \exp(\frac{2r\pi i}{k_i})\exp(\frac{2t\pi i}{k_{n+i}})\Big) \neq 0$. Using the fact that $R(z) = o(|z|)$ we arrive at a contradiction if we choose the polydisk D sufficiently small.

Next, suppose that polynomial map $\pi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is defined by $\pi(z_1, ..., z_{2n}) = (z_1^{k_1}, ..., z_{2n}^{k_{2n}})$. We put

$$X = \{(z_1^{k_1}, ..., z_n^{k_n}, (\overline{z_1} + R_1(z))^{k_{n+1}}, ..., (\overline{z_n} + R_n(z)^{k_{2n}}) : z = (z_1, ..., z_n) \in D\}.$$

Then we have $\pi^{-1}(X) = \cup X_{i_1,...,i_{2n}}$ with

$$X_{i_1,...,i_{2n}} =$$

$$= \{\Big(\rho_{k_1}^{i_1-1} z_1, ..., \rho_{k_n}^{i_n-1} z_n, \rho_{k_{n+1}}^{i_{n+1}-1}(\overline{z}_1 + R_1(z)), ..., \rho_{k_{2n}}^{i_{2n}-1}(\overline{z}_n + R_n(z))\Big) : (z_1, ..., z_n) \in D\}.$$

Therefore by Theorem 1.1

$$P(X_{i_1,...,i_{2n}}) = C(X_{i_1,...,i_{2n}})$$

for all $1 \leqslant i_1 \leqslant k_1$,..., $1 \leqslant i_{2n} \leqslant k_{2n}$. We show that $P(\pi^{-1}(X)) = C(\pi^{-1}(X))$. To do this, we consider the polynomial

$$p(z_1, ..., z_n, z_{n+1}, ..., z_{2n}) = z_1...z_n.z_{n+1}...z_{2n}.$$

We obtain

$$p(X_{i_1,...,i_{2n}}) = \{\rho_{k_1}^{i_1-1}...\rho_{k_{2n}}^{i_{2n}-1}(|z_1|^2 + \overline{z}_1 R_1(z))...(|z_n|^2 + \overline{z}_n R_n(z)) : (z_1, ..., z_n) \in D\}.$$

We see that the sets $p(X_{i_1,...,i_{2n}})$ are contained in an angular sector at 0, situated near the half line through 0 with argument $2\pi(\frac{i_1-1}{k_1} + ... + \frac{i_{2n}-1}{k_{2n}})$. By the comprimeness of k_i, k_j for all $i \neq j$ we obtain

$$\frac{i_1 - 1}{k_1} + ... + \frac{i_{2n} - 1}{k_{2n}} \neq \frac{j_1 - 1}{k_1} + ... + \frac{j_{2n} - 1}{k_{2n}}$$

with $i_s - 1 \neq j_s - 1$ for some $1 \leqslant s \leqslant 2n$. It follows that

$$p(X_{i_1,...,i_{2n}}) \cap p(X_{j_1,...,j_{2n}}) = \{0\}$$

and

$$p^{-1}(0) \cap (X_{i_1,...,i_{2n}} \cup X_{j_1,...,j_{2n}}) = (X_{i_1,...,i_{2n}} \cap X_{j_1,...,j_{2n}}) = (0,...,0)$$

for all $1 \leqslant i_1, j_1, \leqslant k_1$,...,$1 \leqslant i_{2n}, j_{2n}, \leqslant k_{2n}$ with $i_s \neq j_s$ for some $1 \leqslant s \leqslant 2n$.

Furthermore, it is easy to see that $\mathbb{C} \setminus p(X_{i_1,...,i_{2n}})$ is connected set if D suffciently small, so $p(X_{i_1,...,i_{2n}})$ is polynomially convex [3]. Applying Stout's theorem repeatedly, we obtain

$$P(\pi^{-1}(X)) = C(\pi^{-1}(X)).$$

By proposition 2.2 we conclude that $P(X) = C(X)$, or equivalently

$$[z_1^{k_1}, ..., z_n^{k_n}, (\overline{z_1} + R_1)^{k_{n+1}}, ..., (\overline{z_n} + R_n)^{k_{2n}}; D] = C(D).$$

The theorem is proved.

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