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Concept lattice and adjacency matrix

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Abstract. In this paper, we introduce a new encoding for a given binary relation, by using adjacency matrix constructed on the relation. Therefore, a coatom of a concept lattice can be characterized by supports of row vectors of adjacency matrix. Moreover, we are able to compute a poly-sized sub-relation resulting in a sublattice of the original lattice for a given binary relation.

1. Introduction

Lattices have given rise to much interest for the past years, first as a powerful mathematical structure (see e.g. Birkhoff's work from 1967), then as useful in applications such as exploiting questionnaires is Social Sciences (see e.g. Barbet and Monjardet's work from 1970 [1]). Galois lattices were later widely publicized and studied by the large body of work done by Wille and Granter and the many researchers who worked with them, under the name of concept lattices in a much more general context (see e.g. [2]).

Nowadays, concept lattices are well-studied as a classification tool (see [2]), are used in several areas related to Artifical Intelligence and Data Mining, such as Data Base Management, Machine Learning, and Frequent Set Generation (see e.g. [3-5]).

The main drawback of concept lattices is that they may be of exponential size. This makes it impossible, in practice, to compute and span the entire structure they describe. It is thus of primeval importance to be able to navigate the lattice efficiently, or to be able to define a polynomial sized sub-lattice which contains the right information.

In this paper, we introduce a new encoding for a given binary relation, by using adjacency matrix constructed on the relation. Therefore, a coatom of a concept lattice can be characterized by supports of row vectors of adjacency matrix. Moreover, we are able to compute a poly-sized sub-relation resulting in a sublattice of the original lattice for a given binary relation and we used the main results in this paper to determine the concept lattices or a sublattice of given concept lattice which satisfies the above problem.

The paper is organized as follows: Section 2 gives some preliminary notions on concept lattices. In section 3, we give main results.

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2. Preliminaries

In this section, let us recall the notion of concept lattice as far as they are needed for this paper. The definitions in this section are quoted from [5]. A more extensive overview is given in [3]. To allow a mathematical description of extensions and intentions, concept lattice starts with a (formal) context.

Definition 2.1. *A formal context is a triple $K := (G; M; R)$ where G and M are sets and $R \subseteq G \times M$ is a binary relation. The elements of G are called objects and the elements of M attributes. The inclusion $(g; m) \in R$ is read "object g has attribute m". For $A \subseteq G$, we define*

$$A' := \{m \in M | \forall g \in A \ : (g; m) \in R\}$$

and for $B \subseteq M$, we define dually

$$B' := \{g \in G | \forall m \in B \ : (g; m) \in R\}.$$

We assume in this article that all sets are finite, especially G and M. A context K with $|G| = k$ and $|M| = \ell$ is called an k-by-ℓ context. The proofs of the following results are trivial therefore we omit them.

Lemma 2.2. *Let $(G; M; R)$ be a context, $A_1; A_2 \subseteq G$ sets of objects, and $B_1; B_2 \subseteq M$ sets of attributes. Then the following holds:*
(1) $A_1 \subseteq A_2 \Rightarrow A_2' \subseteq A_1'$ and $B_1 \subseteq B_2 \Rightarrow B_2' \subseteq B_1'$.
(2) $A \subseteq A''$ and $B \subseteq B''$.
(3) $A' = A'''$ and $B' = B'''$.
(4) $A \subset B' \Leftrightarrow B \subseteq A' \Leftrightarrow A \times B \subseteq R$.

Definition 2.3. *A formal concept is a pair $(A; B)$ with $A \subseteq G$, $B \subseteq M$, $A' = B$ and $B' = A$. (This is equivalent to $A \subseteq G$ and $B \subseteq M$ being maximal with $A \times B \subset R$.) A is called extent and B is called intent of the concept. The set of all concepts of a formal context K together with the partial order $(A_1; B_1) \leq (A_2; B_2) \Leftrightarrow A_1 \subset A_2$ (which is equivalent to $B_2 \subseteq B_1$) is called concept lattice of K and denote by $\mathcal{L}(R) = \mathcal{L}(G; M; R)$.*

Such a lattice, sometimes refered to as a complete lattice, has a smallest element, called the bottom element, and a greatest element, called the top element.

An element $(A_1; B_1)$ is said to be a predecessor of element $(A; B)$ if $A_1 \subset A$. An element $(A_1; B_1)$ is said to be a ancestor of element $(A; B)$ if $A_1 \subset A$ and there is no intermediate element $(A_2; B_2)$ such that $A_1 \subset A_2 \subset A$. The ancestors of the top element are called coatoms.

Let $K := (G; M; R)$ and $K' := (G'; M'; R')$ be two contexts. We call K and K' isomorphic, and write $K \cong K'$, if there exists two bijections $\varphi : G \to G'$ and $\rho : M \to M'$ such that $(g; m) \in R \Leftrightarrow (\varphi(g); \rho(m)) \in R'$ for all $g \in G$ and $m \in M$.

Theorem 2.4. [The basic theorem of Concept Lattice [5]**]** *The concept lattice of any formal context $(G; M; R)$ is a complete lattice. For an arbitrary set $\{(A_i; B_i) | i \in I\} \subseteq \mathcal{L}(G; M; R)$ of formal concepts, the supremum is given by*

$$\bigvee_{i \in I} (A_i; B_i) = ((\bigcup_{i \in I} A_i)'', \bigcap_{i \in I} B_i)$$

and the infimum is given by

$$\bigwedge_{i\in I}(A_i; B_i) = (\bigcap_{i\in I} A_i, (\bigcup_{i\in I} B_i)'').$$

A complete lattice L is isomorphic to $\mathcal{L}(G; M; R)$ iff there are mappings $\gamma : G \to L$ and $\mu : M \to L$ such that $\gamma(G)$ is supremum-dense and $\mu(M)$ is infimum-dense in L, and

$$gRm \Leftrightarrow \gamma(g) \leq \mu(m).$$

In particular, $L \cong \mathcal{L}(L; L; \leq)$.

The theorem is less complicated as it first may seem (see [5]). We give some explanations below. Readers in a hurry may skip these and continue with the next section.

The first part of the theorem gives the precise formulation for infimum and supremum of arbitrary sets of formal concepts. The second part of the theorem gives (among other information) an answer to the question if concept lattices have any special properties. The answer is "no": every complete lattice is (isomorphic to) a concept lattice. This means that for every complete lattice we must be able to find a set G of objects, a set M of attributes and a suitable relation R, such that the given lattice is isomorphic to $\mathcal{L}(G; M; R)$. The theorem does not only say how this can be done, it describes in fact all possibilities to achieve this.

3. The main results

In the section we assume that $K := (G; M; R)$ is a context with $G = \{g_1, \ldots, g_k\}$ and $M = \{m_1, \ldots, m_\ell\}$. The adjacency matrix $X = (a_{ij})_{\ell\times k}$ of a context $K := (G; M; R)$ is defined by $a_{ij} = 1$ if $(g_j; m_i) \in I$ and $a_{ij} = 0$ otherwise. We denote by X_K the adjacency matrix of a context K. Then we denote by v_i the i^{th} row vector of the adjacency matrix X_K and by $V(K)$ the set of row vectors of the adjacency matrix X_K. For a vector $v = (x_1, \ldots, x_k)$ of $V(K)$, $Supp(v) = \{i \mid x_i = 1\} \subseteq [1, k] = \{1, \ldots, k\}$ and conversion for a subset Z of $[1, k]$, we denote by v_Z the vector in $V(K)$ such that $Z = Supp(v_Z)$. For a subset A of G, we denote by $\overline{A} = \{i \mid g_i \in A\}$ and conversion for a subset Z of $[1, k]$, we denote by A_Z the subset of set G such that $Z = \overline{A_Z}$.

Examble 3.1. Let a binary relation between set $G = \{g_1, g_2, g_3, g_4, g_5\}$ and $M = \{m_1, m_2, m_3, m_4\}$ be the below table. Then the row vector $v_2 = (1, 1, 0, 0, 0)$ and $Supp(v_2) = \{1, 2\}$. Let $Z = \{2, 3, 4\} \subseteq [1, 5]$ then $A_Z = \{g_2, g_3, g_4\}$.

	g_1	g_2	g_3	g_4	g_5
m_1	0	1	1	0	0
m_2	1	1	0	0	0
m_3	1	0	1	1	1
m_4	1	0	0	1	1

Now by Theorem 2.4, every vector of $V(K)$ is attached to a unique concept. Let $K := (G; M; R)$ be some formal context. Then for each vector v of $V(K)$ the corresponding a concept is

$$\varphi(v) := (A''_{Supp(v)}; A'_{Supp(v)}).$$

Lemma 3.2. *Let $K := (G; M; R)$ be a context. Then for all vectors v of $V(K)$,*

$$A''_{Supp(v)} = A_{Supp(v)}.$$

Proof. The inclusion $A_{Supp(v)} \subseteq A''_{Supp(v)}$ is trivial. Assume that $g \in A''_{Supp(v)}$ such that $g \notin A_{Supp(v)}$. Then since $g \in A''_{Supp(v)}$ and $(A''_{Supp(v)}; A'_{Supp(v)})$ is a concept, we have $\{g\} \times A'_{Supp(v)} \subseteq R$. Note that the vector v corresponding with an element m of M and moreover $m \in A'_{Supp(v)}$. Therefore $(g, m) \in R$ and so that $g \in A_{Supp(v)}$, a contradiction. Hence $A''_{Supp(v)} = A_{Supp(v)}$ as required.

Let $v = (x_1, \ldots, x_k)$ and $w = (y_1, \ldots y_k)$ be two vectors in $\mathbb{R}^k$. Then we denote by $v^2 = x_1^2 + \ldots + x_k^2$ and $vw = x_1y_1 + \ldots + x_ky_k$.

Proposition 3.3. *Let X be a subset of coatom of a concept lattice $\mathcal{L}(R)$. Assume that a vector v_i satisfies the condition $v_i^2 = \max\limits_{j=1,\ldots,k}\{v_j^2 \mid Supp(v_j) \not\subseteq In(X) = \bigcup\limits_{(A;B)\in X} \overline{A}\}$. Then the concept $(A; B)$ corresponding with v_i is a coatom of $\mathcal{L}(R)$.* *Proof.* Assume that $(A; B)$ is not a coatom. Then there exists a concept $(A_1; B_1)$ such that $A \subset A_1$. Let $m_t \in B_1$. Since $(A_1; B_1)$ is a concept, we get that $A_1 \times \{m_t\} \subseteq R$. Then $A_1 \subset A_{Supp(v_t)}$ and so that $\overline{A} = Supp(v_i) \subset Supp(v_t)$. Since $Supp(v_i) \not\subseteq In(X)$, we have $Supp(v_t) \not\subseteq In(X)$. Hence, $v_i^2 < v_t^2$ and $Supp(v_t) \not\subseteq In(X)$ in contradiction by $v_i^2 = \max\limits_{j=1,\ldots,k}\{v_j^2 \mid Supp(v_j) \not\subseteq In(X)\}$. Thus $(A; B)$ is a coatom of $\mathcal{L}(R)$.

Theorem 3.4. *We use the above notation. Then the following two statements are equivalent.*

(i) A concept $(A; B)$ is a coatom of $\mathcal{L}(R)$.

(ii) Vector $v = v_{\overline{A}}$ satisfies the condition $Supp(v) \not\subseteq Supp(v_i)$ for all vectors v_i such that $v_i^2 > v^2$.

Proof. (i)$\Rightarrow$ (ii) A concept $(A; B)$ is a coatom. Let $v = v_{\overline{A}}$. Then $A \not\subseteq A_1$ for all $A_1 \neq \emptyset$ and A_1 is a extent of any concept. By Lemma , if a vector v_i satisfies $v_i^2 > v^2$, then $Supp(v_i) \not\subseteq Supp(v)$.

(ii)$\Rightarrow$ (i) Let v be a vector such that $Supp(v) \not\subseteq Supp(v_i)$ where a vectors v_i satisfies $v^2 < v_i^2$. Assume that a concept $(A; B)$ where $A = A_{Supp(v)}$ is not a coatom. Then there exists a concept $(A_1; B_1)$ such that $A \subset A_1$. Let $m_t \in B_1$. Since $A_1 \times B_1$ is a concept, we have $A_1 \times \{m_t\} \subseteq R$. Therefore $A_1 \subseteq A_{Supp(v_t)}$. Then we obtain $Supp(v) \subset Supp(v_t)$, and so that $v^2 < v_t^2$, a contradiction. Hence $(A; B)$ is a coatom.

Corollary 3.5. *Let $(A; B)$ be a coatom of lattice $\mathcal{L}(R)$. Then we have*

$$v_{\overline{A}}^2 = \max\{v^2 \mid Supp(v) \not\subseteq Supp(v_i) \text{ for all } v_i^2 > v_{\overline{A}}^2 \text{ and } v \in V(K)\}.$$

Proof. Put $C = \{v \mid Supp(v) \not\subseteq Supp(v_i) \text{ for all } v_i^2 > v_{\overline{A}}^2; v \in V(K)\}$. Since $(A; B)$ is a coatom by Theorem , we obtain $Supp(v_{\overline{A}}) \not\subseteq Supp(v_i)$ where a vector v_i satisfies $v_i^2 > v_{\overline{A}}^2$. Therefore $v_{\overline{A}}^2 \leq \max\limits_{v\in B} v^2$. For all $v \in C$, $v^2 \leq v_{\overline{A}}^2$, we have $\max\limits_{v\in B} v^2 \leq v_{\overline{A}}^2$. Hence $v_{\overline{A}}^2 = \max\{v^2 \mid Supp(v) \not\subseteq Supp(v_i) \text{ for all } v_i^2 > v_{\overline{A}}^2 \text{ and } v \in V(K)\}$, as required.

Note that a vector $v \in V(K)$ corresponds with a concept which is coatom or without. Moreover, two vectors v_i and v_j are different but they correspond with a same concept.

Corollary 3.6. *Let v and w be two vectors in $V(K)$ such that $A_{Supp(v)}$ and $A_{Supp(w)}$ are two extents of any coatoms. Then the following two statements are equivalent.*

(i) Vectors v and w correspond with a same coatom.

(ii) $Supp(v) = Supp(w)$.

(iii) $v^2 = w^2 = vw$.

Proof. (i)$\Leftrightarrow$ (ii) and (ii)$\Rightarrow$ (iii) are trivial.

(iii)$\Rightarrow$ (i): Since entries of vectors v and w are 0 or 1 if $Supp(v) \not\subseteq Supp(w)$ then $v^2 > vw$. Therefore $Supp(v) = Supp(w)$.

Let $V(R) = \{v \in V(K) \mid v^2 = \max_{i\in[1,k]} v_i^2\}$ and $X_R = \{(A_{Supp(v)}; A'_{Supp(v)}) | v \in V(R)\}$.

Corollary 3.7. *A set X_R is a subset of coatoms of the lattice $\mathcal{L}(R)$.*
Proof. Let $(A; B) \in X_R$. Then a vector $v_{\overline{A}}$ satisfies the condition $v_{\overline{A}}^2 = \max_{i\in[1,k]} v_i^2\}$, and thus there dosen't exists a vector w such that $w^2 > v_{\overline{A}}^2$. By Theorem , a concept $(A; B)$ is a coatom as required.
Example 3.8. Let $K = (G; M; R)$ be as in Example . Then we have $v_4 = (1, 0, 0, 1, 1)$ and so that $\varphi(v_4) = (\{g_1, g_4, g_5\}; \{m_3, m_4\})$ is a concept of lattice $\mathcal{L}(R)$ by Lemma . Moreover, we have $v_1^2 = v_2^2 = 2$, $v_3^2 = 4$ and $v_4^2 = 3$. Then by Theorem , we get that $\varphi(v_4)$ is not a coatom of this lattice since $Supp(v_4) \subset Supp(v_3)$. On the other hand, $\varphi(v_2) = (\{g_1, g_2\}; \{m_2\})$ is a coatom because $Supp(v_2) \nsubseteq Supp(v_3)$ and $Supp(v_2) \nsubseteq Supp(v_4)$.

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