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Characterized rings by pseudo - injective modules

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Abstract: It is shown that:
(1) Let R be a simple right Noetherian ring, then the following conditions are equivalent:
(i) R is a right SI ring;
(ii) Every cyclic singular right R - module is pseudo - injective.
(2) Let R be a right artinian ring such that every finite generated right R - module is a direct sum of a projective module and a pseudo - injective module. Then:
(i) $R/Soc(R_R)$ is a semisimple artinian ring;
(ii) $J(R) \subset Soc(R_R)$;
(iii) $J^2(R) = 0$.
(3) Let R be a ring with condition $(^*)$, then every singular right R - module is isomorphic with a direct sum of pseudo - injective modules.

1. Introduction

Throughout this note, all rings are associative with identity, and all modules are unital right modules. The socle and the Jacobson radical of M are denoted by $Soc(M)$ and $J(M)$. Given two R - modules M and N, N is called M - injective if for every submodule A of M, any homomorphism $\alpha : A \longrightarrow N$ can be extended to a homomorphism $\beta : M \longrightarrow N$. A module$N$ is called injective if it is M - injective for every R - module M. On the other hand, N is called quasi - injective if N is N - injective. For basic properties of injective modules we refer to [1-4].

We say N is M - pseudo - injective (or pseudo - injective relative to M) if for every submodule X of M, any monomorphism $\alpha : A \longrightarrow N$ can be extended to a homomorphism $\beta : M \longrightarrow N$. N is called pseudo - injective if N is N - pseudo - injective. We have the following implications:

Injective $\Rightarrow$ quasi - injective $\Rightarrow$ pseudo - injective.

We refer to [5-8] for background on pseudo - injective modules.

Let M be a module. A module $Z(M)$ is called singular submodule of M if $Z(M) = \{x \in M \mid xI = 0$ for some essential right ideal of $R\}$. If $Z(M) = M$ then M is called singular module, while if $Z(A) = 0$

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then A is called nonsingular module. A ring R is called a right (left) SI ring if every singular right R - module is injective. For basic properties of singular (nonsingular) modules and SI rings we refer to [3] and [9].

For a ring R consider the following conditions:

$(^*)$ Every cyclic right R - module is a direct sum of a projective module and a pseudo - injective module.

$(^{**})$ The direct sum of every family of pseudo - injective right R - modules is also pseudo - injective.

$(^{***})$ Every singular right R - module is a pseudo - injective module.

In [10, Theorem 1], it was shown that a simple ring R is right SI iff every cyclic singular right $R-$module is quasi- continuous. In this paper, we give characteristics of SI rings by class modules pseudo - injective. We prove that a simple ring R is right SI iff every cyclic singular right R - module is pseudo - injective. Note that, every quasi - injective modules is quasi- continuous and pseudo - injective module; but pseudo - injective module is not quasi- continuous and quasi - continuous module is not pseudo - injective. We give also characteristics of artinian rings by class modules pseudo - injective

2. The results

Theorem 2.1. *Let R be a simple right Noetherian ring, then the following conditions are equivalent:*

(i) R is a right SI ring;

(ii) Every cyclic singular right R - module is pseudo injective.

Proof. $(i) \Longrightarrow (ii)$ is clear.

$(ii) \Longrightarrow (i)$. Let R be a simple right Noetherian ring whose cyclic singular right $R-$ modules are pseudo injective. We show that R is a right SI ring. If $Soc(R_R) \neq 0$, then $Soc(R_R) = R$, and hence R is a simple artinian ring. We are done. Next consider the case $Soc(R_R) = 0$. We prove that any artinian right R - module A is semisimple.

Assume that $A \neq 0$, we imply $A \cong F/K$ with F is a free module. We show that K is an essential submodule of F. Assume K is not essential in F, there is a submodule T of F such that $K \cap T = 0$, i.e., $K \oplus T$ is a submodule of F. Note that $A \cong F/K \supset (K \oplus T)/K \cong T$. Since A is a artinian module, thus $Soc(T) \neq 0$. Hence $Soc(F) \neq 0$. But $F = \oplus_{i\in I} R_i$ with $R_i \cong R_R$ for any $i \in I$. We imply $Soc(F) = \oplus_{i\in I} Soc(R_i) = 0$, a contradiction. Therefore K is an essential submodule of F, i.e., A is a singular module. Let X be a cyclic submodule of A. We have $Soc(X) \neq 0$. By R is right noetherian, and X is a finitely generated module, we imply $Soc(X) = X_1 \oplus ... \oplus X_k$ where each X_t is simple module. By [11, Lemma 3.1], we can show that $X \oplus X_1$ is cyclic. By induction, we have $X \oplus Soc(X)$ is cyclic. Hence $X \oplus Soc(X)$ is pseudo injective. By [7, Theorem 2.2] or [5, Corollary 5], thus $Soc(X)$ is X - injective. Therefore $Soc(X)$ is a direct summand of X. Since X is a artinian module, we imply $X = Soc(X) \subseteq Soc(A)$. This shows that A is semisimple.

Now, we prove that every singular cyclic module over R is semisimple, or equivalently, for each essential right ideal C of R, R/C is semisimple. By the above claim, it suffices to that show R/C is artinian. Hence R is a SI ring.

Assume on the contrary that there is an essential right ideal A of R such that R/A is not artinian. Since R is right noetherian, there exists an essential right ideal L of R which is maximal with respect to the condition that $M = R/L$ is not artinian. We show that M is uniform and $Soc(M) = 0$. Assume M is not uniform, there are two submodules P_1, P_2 of R_R such that $L \subset P_1, P_2 \subset R_R$, $L \neq P_1$, $L \neq P_2$ and $P_1 \cap P_2 = L$. Let f be a homomorphism $R \longrightarrow R/P_1 \oplus R/P_2$, $f(r) = (r + P_1, r + P_2)$

then, $Kerf = P_1 \cap P_2 = L$. We have $M = R/L = R/Kerf \cong Imf \subseteq R/P_1 \oplus R/P_2$. Note that, R/P_1 and R/P_2 are artinian, thus R/L is also artinian, a contradiction. Hence M is uniform. By M is not a simple module, we imply $Soc(M) = 0$. Moreover, for any nonzero submodule N of M, we have $N = N_1/L$ with $L \subset N_1$ and $L \neq N_1$. By $M/N = R/L/N_1/L \cong R/N_1$, thus M/N is a artinian module. Therefore M/N is a semisimple module. Let U and V be submodules of M with $0 \neq U \subset V \subset M$ and $U \neq V \neq M$. Then V/U is a submodule of semisimple artinian module M/U. Hence V/U is semisimple artinian module. We imply $V/U = S_1 \oplus ... \oplus S_t$, where each S_j is simple module. Consider the module $Q = M \oplus V$. Since M is cyclic and $Q/(0 \oplus U) \cong M \oplus (V/U) = M \oplus (S_1 \oplus ... \oplus S_t)$. By [11, Lemma 3.1], we can show that $M \oplus S_1$ is cyclic. By induction, we have $M \oplus (S_1 \oplus ... \oplus S_t)$ is cyclic, i.e., $Q/(0 \oplus U)$ is cyclic. By [10], we can choose $x \in Q$ such that $[xR + (0 \oplus U)]/(0 \oplus U) = Q/(0 \oplus U)$ and xR contains $M \oplus 0$. Note that xR is not uniform. By modularity $xR = xR \cap Q = xR \cap (M \oplus V) = M \oplus W$ where $(0, W) = xR \cap (0, V) \neq (0, 0)$. By M is a singular module, thus Q is also a singular module. Hence xR is a singular cyclic module. Since xR is a pseudo - injective module, i.e., $M \oplus W$ is pseudo - injective. By [7, Theorem 2.2], W is M- injective. Therefore W is a direct summand of M, a contradiction. Hence R/C is artinian for every essential right ideal C of R. Thus R/C is a semisimple module, i.e., R is a right SI ring.

Theorem 2.2. *Let R be a right artinian ring such that every finite generated right R - module is a direct sum of a projective module and a pseudo - injective module. Then:*

(i) $R/Soc(R_R)$ is a semisimple artinian ring;

(ii) $J(R) \subset Soc(R_R)$;

(iii) $J^2(R) = 0$.

Proof. (i) Set $A = R/Soc(R_R)$. By R is a right artinian ring, thus A is a singular right R - module. Set $M = A \oplus Soc(A)$, then M is a singular module. Since M is a finite generated right R - module, we imply $M = X \oplus Y$ where X is a pseudo - injective module and Y is a projective module. Note that Y is a singular module, we have $Y \cong F/K$ with K is an essential submodule of F. By Y is a projective module, we imply K is a direct summand of F. Hence $K = F$, i.e., $Y = 0$. Therefore $M = A \oplus Soc(A)$ is a pseudo - injective module. By [7, Theorem 2.2], $Soc(A)$ is $M-$ injective. By A is artinian, thus A is semisimple artinian ring, proving (i).

(ii) and (iii). By properties (i).

Theorem 2.3. *Let R be a ring with condition $(^*)$, then every singular right R - module is isomorphic with a direct sum of pseudo - injective modules.*

Proof. Let R be a ring with condition $(^*)$. Let X be a cyclic singular right R - module, then by condition $(^*)$ we have $X = P \oplus A$ with P is a projective and A is a pseudo - injective. Since P is singular module, thus $P \cong U/V$ with V is an essential submodule of U. By P is a projective module, we imply V is a direct summand of U. Hence $U = V$, i.e., $P = 0$. Therefore X is a pseudo - injective module.

Finally, if M is a singular right R- module, then $M \cong F/K$ with F is a free module and K is an essential submodule of F. Note that $F \cong \oplus_{i \in I} R_i$ with $R_i \cong R_R$, we have $M \cong F/K = (\oplus_{i \in I} R_i)/K \cong \oplus_{i \in I}(R_i + K)/K$. By $(R_i + K)/K$ is a cyclic singular module, thus it is a pseudo - injective module. Therefore, M is isomorphic with a direct sum of pseudo - injective modules.

Corollary 2.4. *Let R be a ring satisfies conditions $(^*)$ and $(^{**})$, then R satisfies condition $(^{***})$.*

Proof. By condition $(^{**})$ and Theorem 2.3.

3. Examples

1) ([8, page 364]) Let $F = \mathbf{Z}_2$ where $\mathbf{Z}$ is the ring of integer numbers and $A = F[X]$. Then $A/(x)$ is a $(A/(x) - A/(x^2))-$bimodule in the natural way, and

$$R = \{\begin{pmatrix} u & 0 \\ v & w \end{pmatrix} \mid u, v \in A/(x), w \in A/(x^2)\}$$

is a ring with usual binary operations. Let M be the right ideal

$$M = \{\begin{pmatrix} 0 & 0 \\ v & w \end{pmatrix} \mid v \in A/(x), w \in A/(x^2)\}.$$

Then M_R is pseudo - injective but not quasi - injective.

2) Consider the following ring

$$R = \begin{pmatrix} \mathbf{C} & \mathbf{C} \\ 0 & \mathbf{R} \end{pmatrix}$$

where $\mathbf{R}$ and $\mathbf{C}$ are the fields of real and complex numbers, respectively. By [12, page 152], every right $R-$modules is a direct sum of a projective module and a quasi - injective module. Hence R satifies conditions $(^*)$ and $(^{***})$.

3) Consider the following ring

$$R = \begin{pmatrix} \mathbf{R} & \mathbf{C} \\ 0 & \mathbf{C} \end{pmatrix}$$

where $\mathbf{R}$ and $\mathbf{C}$ are the fields of real and complex numbers, respectively. By [12, page 151], every right $R-$modules is a direct sum of a projective module and a quasi - injective module. Hence R satifies conditions $(^*)$ and $(^{***})$. By [12, page 151], R is right SI ring.

4) Consider the following ring

$$R = \begin{pmatrix} \mathbf{R} & \mathbf{R} \\ 0 & \mathbf{R} \end{pmatrix}$$

where $\mathbf{R}$ is the field of real numbers. By [13, Remark 3.4 (page 464)], R is a $CS-$semisimple ring. Hence R is a right and left Artinian ring (see [11, Theorem 1.1]). Let $M_i, \forall i \in I$ be pseudo - injective right $R-$modules, then M_i is $CS-$module where each $i \in I$. Hence M_i is quasi - injective (see [5, Theorem 5]). By [4], $M = \oplus_{i\in I} M_i$ is a quasi - injective module. Hence M is a pseudo - injective module, i.e., R satifies conditions $(^{**})$.

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