GENERALIED MASON'S THEOREM

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Abstract. The purpose of this paper is to give a generalization of Mason's theorem by the Wronskian technique over fields of characteristic 0.
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1. Introduction

Let F be a fixed algebraically closed field of characteristic 0. Let $f(z)$ be *a polynomial non - constants* which coefficients in F and let $\overline{n}(1/f)$ be the number of distinct zeros of f. Then we have the following.

Marson's theorem. ([2]). *Let $a(z), b(z), c(z)$ be relatively prime polynomials in F and not all constants such that $a+b=c$. Then*

$$\max\{\deg(a), \deg(b), \deg(c)\} \leqslant \overline{n}\left(\frac{1}{abc}\right) - 1.$$

It is now well known that Mason's Theorem implies the following corollary.

Corollary. (Fermat's Theorem over polynomials). *The equation $x^n + y^n = z^n$ has no solutions in non - constants and relatively prime polynomials in F if $n \leqslant 3$.*

The main theorem in this paper is as following:

Theorem 1.1. *Les $f_0, f_1, ..., f_n$ be relatively primer polynomials and $f_0, f_1, ..., f_n$ be linearly independent over F. If*

$$f_0 + f_1 + ... + f_n = f_{n+1},$$

then

$$\max_{0 \leqslant i \leqslant n+1} \deg f_i \leqslant n\left(\sum_{i=0}^{n+1} \overline{n}\left(\frac{1}{f_i}\right)\right) - \frac{n(n+1)}{2}.$$

Remark. Theorem 1.1 is a generalization of Mason's theorem which was obtained for case $n = 1$.

2. Proof of the main theorem

Let $\varphi(x) = \frac{f(x)}{g(x)} \not\equiv 0$ be a rational function, where $f(x), g(x)$ are non - zero and relatively prime polynomials on F. The degree of $\varphi(x)$, denoted by $\deg \varphi(x)$, is defined to be $\deg f(x) - \deg g(x)$. Here the notation $\deg f(x)$ means the degree of polynomial $f(x)$.

From the properties of polynomial, we have.

Proposition 2.1. *If φ_1 and φ_2 are the rational functions on F, then*

1) $\deg(\varphi_1\varphi_2) = \deg \varphi_1 + \deg \varphi_2$

2) $\deg \left(\dfrac{1}{\varphi_1}\right) = -\deg \varphi_2$

3) $\deg(\varphi_1 + \varphi_2) \leqslant \max(\deg \varphi_1, \deg \varphi_2)$.

Definition 2.2. Let $\varphi(x) \not\equiv 0$ be a rational function on F. For every $a \in F$, we write

$$\varphi(x) = (x-\alpha)^m \frac{f_1(x)}{g_1(x)}, (m \in \mathbb{Z}),$$

where $f_1(x), f_2(x)$ are relatively prime polynomials and $f_1(\alpha) \neq 0, g_1(\alpha) \neq 0$. We call m *order* of φ at α.

Proposition 2.3. *If φ_1, φ_2 are rational functions on F and $a \in F$, then*

1) $ord_\alpha(\varphi_1\varphi_2) = ord_\alpha\varphi_1 + ord_\alpha\varphi_2$

2) $ord_\alpha(\frac{1}{\varphi_1}) = -ord_\alpha\varphi_1$

3) $ord_\alpha(\frac{\varphi_1}{\varphi_2}) = ord_\alpha\varphi_1 - ord_\alpha\varphi_2$.

Proposition 2.4. *Let $\varphi(x)$ be a the rational function on F and let the derivatives order k, $\varphi^{(k)} \not\equiv 0$. Then*

$$ord_\alpha \left(\frac{\varphi^{(k)}}{\varphi}\right) \geqslant -k.$$

Proof. Let $\varphi(x) = (x-\alpha)^m \frac{f(x)}{g(x)}$, where $f(x), g(x)$ are relatively prime and $f(\alpha)g(\alpha) \neq 0$. Then, we have

$$\varphi'(x) = (x-\alpha)^{m-1} \frac{(mf(x) + (x-\alpha)f'(x)) + (x-\alpha)f(x)g'(x)}{g^2(x)}.$$

Since $ord_\alpha(g(x)) = 0$, we have

$$ord_\alpha(\varphi'(x)) \geqslant m - 1.$$

Therefore

$$ord_\alpha \left(\frac{\varphi'}{\varphi}\right) = ord_\alpha(\varphi') - ord_\alpha(\varphi) \geqslant -1.$$

Thus, we obtain

$$
\begin{aligned}
ord_\alpha\left(\frac{\varphi^{(k)}}{\varphi}\right) &= ord_\alpha\left(\frac{\varphi'}{\varphi}.\frac{\varphi''}{\varphi'}\cdots\frac{\varphi^{(k)}}{\varphi^{(k-1)}}\right)\\
&= ord_\alpha\left(\frac{\varphi'}{\varphi}\right) + ord_\alpha\left(\frac{\varphi''}{\varphi'}\right) + \ldots + ord_\alpha\left(\frac{\varphi^{(k)}}{\varphi^{(k-1)}}\right) \geqslant -k \qquad (1)
\end{aligned}
$$

Proposition 2.5. *Let φ_1, φ_2 be rational functions on F and $a \in F$. Then*

$$ord_\alpha(\varphi_1, \varphi_2) \geqslant min\{ord_\alpha\varphi_1, ord_\alpha\varphi_2\}\ .$$

Proof. Let $ord_\alpha\varphi_1 = m_1$ and $ord_\alpha\varphi_2 = m_2$. Then

$$\varphi_1(x) = (x-\alpha)^m \frac{f_1(x)}{g_1(x),} \qquad (2)$$

$$\varphi_2(x) = (x-\alpha)^m \frac{f_2(x)}{g_2(x),} \qquad (3)$$

where f_1, f_2, g_1, g_2 are the polynomials over F and $f_1(\alpha), f_2(\alpha), g_1(\alpha), g_2(\alpha) \neq 0$. We set $m = min(m_1, m_2)$. Then

$$\varphi_1(x) + \varphi_2(x) = (x-\alpha)^m \frac{\left[(x-\alpha)^{m_1-m} f_1(x)g_2(x) + (x-\alpha)^{m_2-m} f_2(x)g_1(x)\right]}{f_2(x)g_2(x)}.$$

Since $f_2(\alpha)g_2(\alpha) \neq 0$, we have

$$ord_\alpha(\varphi_1 + \varphi_2) \geqslant m = min(ord_\alpha\varphi_1, ord_\alpha\varphi_2).$$

Definition 2.6. Let $f_1, f_2, ..., f_n$ be polynomials on F (but to a large extent what we do depends only on formal properties of devivations). We recall that their $Wronskian$ is

$$W(f_1, f_2, ..., f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Remark. If $f_1, f_2, ..., f_n$ are linearly independent on F, then $W(f_1, f_2, ..., f_n) \neq 0$.

Proof of Theorem 1.1. Let $\{\alpha_0, \alpha_1, ..., \alpha_n\}$ be a subset of $I = \{0, 1, ..., n+1\}$. Then the equation $f_0 + f_1 + ... + f_n = f_{n+1}$ implies $W(f_{\alpha_0}, ..., f_{\alpha_n}) = \delta W(f_0, f_1, ..., f_n)$, where $\delta = 1$ or -1. Because $f_0, f_1, ..f_n$ are linearly independent, we obtain

$$W(f_0, f_1, ..., f_n) \neq 0.$$

Then, we set

$$P(t)=\frac{W(f_0,f_1,...,f_n)}{f_0f_1....f_n},$$

$$Q(t)=\frac{f_0f_1....f_{n+1}}{W(f_0,f_1,...,f_n)}.$$

Hence, we have

$$f_{n+1}=P(t)Q(t).$$

We first prove that

$$degQ(t)\leqslant n\left(\sum_{i=0}^{n+1}\overline{n}\left(\frac{1}{f_i}\right)\right).$$

Let α be a zero of the function $Q(z)$. Then α is a zero of some polynomial $f_i(0\leqslant i\leqslant n+1)$. By the hypothesis that the polynomials are relatively prime, there exists a number $v(0\leqslant v\leqslant n+1)$ such that $f_v(\alpha)\neq 0$.

Let $\{i_0,i_1,...,i_n\}$ be a subset $I|\{v\}$, then we have

$$Q(t)=\delta\frac{f_{i_0}f_{i_1}....f_{i_n}}{W(f_0,f_1,...,f_n)}f_v.$$

Denote

$$R(t)=\frac{W(f_{i_0},f_{i_1},....,f_{i_n})}{f_{i_0}f_{i_1}...f_{i_n}}$$

as the logarithmic Wronskian corresponding to $\{i_0,i_1,....,i_n\}$, which is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{f'_{i_0}}{f_{i_0}} & \frac{f'_{i_1}}{f_{i_1}} & \cdots & \frac{f'_{i_n}}{f_{i_n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{f^{(n-1)}_{i_0}}{f_{i_0}} & \frac{f^{(n-1)}_{i_1}}{f_{i_1}} & \cdots & \frac{f^{(n-1)}_{i_n}}{f_{i_n}} \end{vmatrix}$$

Then $f_v=R(t)Q(t)$ and so $ord_\alpha R(t)=-ord_\alpha Q(t)$. Then the determinant $R(t)$ is a sum of following terms

$$\delta\frac{f'_{\alpha_0}f''_{\alpha_1}....f^{(n-1)}_{\alpha_n}}{f_{\alpha_0}f_{\alpha_1}...f_{\alpha_n}},$$

where $0\leqslant\alpha_0,\alpha_1,...,\alpha_n\leqslant n+1$ and $\delta=1$ or -1.

By applying the propositions 2.3 and 2.4, we get

$$\begin{aligned} ord_\alpha\left(\frac{f'_{\alpha_0}f''_{\alpha_1}...f^{(n-1)}_{\alpha_n}}{f_{\alpha_0}f_{\alpha_1}...f_{\alpha_n}}\right) &= ord_\alpha\left(\frac{f'_{\alpha_0}}{f_{\alpha_0}}\right)+ord_\alpha\left(\frac{f''_{\alpha_1}}{f_{\alpha_1}}\right)+...+ord_\alpha\left(\frac{f^{(n-1)}_{\alpha_n}}{f_{\alpha_n}}\right) \\ &\geqslant -n\left(\sum_{\substack{0\leqslant i\leqslant n+1\\ f_i(a)=0}}1\right). \end{aligned} \tag{4}$$

Therefore from Proposition 2.5, we have

$$ord_\alpha R(t) \leqslant -n\left(\sum_{\substack{0\leqslant i\leqslant n+1\\ f_i(a)=0}} 1\right)$$

and so

$$ord_\alpha Q(t) = -ord_\alpha R(t) \leqslant -n\left(\sum_{\substack{0\leqslant i\leqslant n+1\\ f_i(a)=0}} 1\right).$$

Since this inequality holds for any zero α of $Q(t)$, we get

$$\deg Q(t) \leqslant n\left(\sum_{i=0}^{n+1} \overline{n}\left(\frac{1}{f_i}\right)\right).$$

Next, we will prove that

$$\deg P(t) \leqslant -\frac{n(n+1)}{2}.$$

Here, we have $P(t)$ as the logarithmic Wronskian corresponding to $I = \{0, 1, .., n\}$ which is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{f_0'}{f_0} & \frac{f_1'}{f_1} & \cdots & \frac{f_n'}{f_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{f_0^{(n)}}{f_0} & \frac{f_1^{(n)}}{f_1} & \cdots & \frac{f_n^{(n)}}{f_n} \end{vmatrix}$$

The determinant $P(t)$ is a sum of following terms

$$\delta\frac{f_{\beta_0}' f_{\beta_1}'' \ldots . f_{\beta_n}^{(n-1)}}{f_{\beta_0} f_{\beta_1} \ldots f_{\beta_n}}.$$

For every term, by Proposition 2.4 we have

$$\begin{aligned}\deg\left(\frac{f_{\beta_0}' f_{\beta_1}'' \ldots f_{\beta_n}^{(n-1)}}{f_{\beta_0} f_{\beta_1} \ldots f_{\beta_n}}\right) &= \deg\left(\frac{f_{\beta_0}'}{f_{\beta_0}}\right)\deg\left(\frac{f_{\beta_1}''}{f_{\beta_1}}\right) + \ldots + \deg\left(\frac{f_{\beta_n}^{(n)}}{f_{\beta_n}}\right) \\ &= -(1+2+\ldots+n) = -\frac{n(n+1)}{2}. \qquad (5)\end{aligned}$$

Therefore

$$\deg P(t) \leqslant -\frac{n(n+1)}{2}$$

so

$$\deg f_{n+1} = \deg P(t) + \deg Q(t) \leqslant n\left(\sum_{i=0}^{n+1} \overline{n}\left(\frac{1}{f_i}\right)\right) - \frac{n(n+1)}{2}.$$

By the similar arguments applying to the polynomial $f_0, f_1, ..., f_n$, we have

$$\max_{0 \leqslant \subset \leqslant n+1} (deg f_i) \leqslant n \left(\sum_{i=0}^{n+1} \overline{n} \left(\frac{1}{f_i} \right) \right) - \frac{n(n+1)}{2}.$$

Theorem 1.1 is proved.

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