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THE HYPERSURFACE SECTIONS AND POINTS IN UNIFORM POSITION

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Abstract. The aim of this paper is to show that the preservation of irreducibility of sections between a variety and hypersurface by specializations and almost all sections between a linear subspace of dimension $h = n - d$ of $\mathbb{P}_k^n$ and a nondegenerate variety of dimension $d > 0$ consists of s points in uniform position.

Introduction

The lemma of Haaris [2] about a set in the uniform position has attracted much attention in algebraic geometry. That is a set of points of a projective space such that any two subsets of them with the same cardinality have the same Hilbert function. For wider applicability of the result, in this paper we will now apply this lemma to prove that almost all $n-d$-dimensional linear subspace sections of a d-dimensional irreducible nondegenerate variety in $\mathbb{P}^n$ are the finite sets of points in uniform position under certain conditions. Here we use a notion *ground-form* which was given by E. Noether, see [3] or [6], and specializations of ideals and of modules [3], [4], [5], [6], [7], that is a technique to prove the existence of algebraic structures over a field with prescribed properties.

Let k be an infinite field of arbitrary characteristic. Let $u = (u_1, \dots, u_m)$ be a family of indeterminates and $\alpha = (\alpha_1, \dots, \alpha_m)$ a family of elements of k. We denote the polynomial rings in n variables $x_1, \dots, x_n$ over $k(u)$ and $k(\alpha)$ by $R = k(u)[x]$ and by $R_\alpha = k(\alpha)[x]$, respectively. The theory of specialization of ideals was introduced by W. Krull [3]. Let I be an ideal of R. A specialization of I with respect to the substitution $u \to \alpha$ was defined as the ideal $I_\alpha = \{f(\alpha, x) |\ f(u, x) \in I \cap k[u, x]\}$. For almost all the substitutions $u \to \alpha$, that is for all α lying outside a proper algebraic subvariety of k^m, specializations preserve basic properties and operations on ideals, and the ideal I_α inherits most of the basic properties of I. Specializations of finitely generated modules M_u over $R_u = k(u)[x]$, one can substitute u by a finite set α of elements of k to obtain the modules M_α over $R = k[x]$ with a same properties [4], and specializations of finitely generated graded modules over the graded ring $R_u = k(u)[x]$ are also graded [5]. The interested reader is referred to [5] for more details. Using the notion of Ground-form of an unmixed ideal and results in the specializations of graded modules we will prove

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX preservation of irreducibility of hypersurface sections and apply a lemma of Harris to give some properties about set of points on a variety.

In this paper we shall say that a property holds for almost all α if it holds for all points of a Zariski-open non-empty subset of k^m. For convenience we shall often omit the phrase "for almost all α" in the proofs of the results of this paper.

1. Some results about specializations of graded modules

We shall begin with recalling the specializations of finitely generated graded modules.

Let k be an infinite field of arbitrary characteristic. Let $u = (u_1, \ldots, u_m)$ be a family of indeterminates and $\alpha = (\alpha_1, \ldots, \alpha_m)$ a family of elements of k. To simplify notations, we shall denote the polynomial rings in $n+1$ variables $x_0, \ldots, x_n$ over $k(u)$ and $k(\alpha)$ by $R = k(u)[x]$ and by $R_\alpha = k(\alpha)[x]$, respectively. The maximal graded ideals of R and R_α will be denoted by $\mathfrak{m}$ and $\mathfrak{m}_\alpha$. It is well-known that each element $a(u,x)$ of R can be written in the form

$$a(u,x) = \frac{p(u,x)}{q(u)}$$

with $p(u,x) \in k[u,x]$ and $q(u) \in k[u] \setminus \{0\}$. For any α such that $q(\alpha) \neq 0$ we define

$$a(\alpha,x) = \frac{p(\alpha,x)}{q(\alpha)}.$$

Let I is an ideal of R. Following [3], [7] we define the specialization of I with respect to the substitution $u \to \alpha$ as the ideal I_α of R_α generated by elements of the set

$$\{f(\alpha,x) |\ f(u,x) \in I \cap k[u,x]\}.$$

For almost all the substitutions $u \to \alpha$, specializations preserve basic properties and operations on ideals, and the ideal I_α inherits most of the basic properties of I, see [3].

The specialization of a free R-module F of finite rank is a free R_α-module F_α of the same rank as F. Let $\phi : F \longrightarrow G$ be a homomorphism of free R-modules. We can represent ϕ by a matrix $A = (a_{ij}(u,x))$ with respect to fixed bases of F and G. Set $A_\alpha = (a_{ij}(\alpha,x))$. Then A_α is well-defined for almost all α. The specialization $\phi_\alpha : F_\alpha \longrightarrow G_\alpha$ of ϕ is given by the matrix A_α provided that A_α is well-defined. We note that the definition of ϕ_α depends on the chosen bases of F_α and G_α.

Definition. [4] Let L be a finitely generated R-module. Let $F_1 \xrightarrow{\phi} F_0 \longrightarrow L \longrightarrow 0$ be a finite free presentation of L. Let $\phi_\alpha : (F_1)_\alpha \longrightarrow (F_0)_\alpha$ be a specialization of ϕ. We call $L_\alpha := \operatorname{Coker} \phi_\alpha$ a *specialization* of L (with respect to ϕ).
It is well known [4, Proposition 2.2] that L_α is uniquely determined up to isomorphisms.

Lemma 1.1. [4, Theorem 3.4] *Let L be a finitely generated R-module. Then there is* $\dim L_\alpha = \dim L$ *for almost all* α.

Let R be naturally graded. For a finitely generated graded R-module L, we denote by L_t the homogeneous component of L of degree t. For an integer h we let $L(h)$ be the same module as L with grading shifted by h, that is, we set $L(h)_t = L_{h+t}$.

Let $F = \bigoplus_{j=1}^{s} R(-h_j)$ be a free graded R-module. We make the specialization F_α of F a free graded R_α-module by setting $F_\alpha = \bigoplus_{j=1}^{s} R_\alpha(-h_j)$. Let $\phi : \bigoplus_{j=1}^{s_1} R(-h_{1j}) \longrightarrow \bigoplus_{j=1}^{s_0} R(-h_{0j})$ be a graded homomorphism of degree 0 given by a homogeneous matrix $A = (a_{ij}(u,x))$, where all $a_{ij}(u,x)$ are the forms with

$$\deg a_{ij}(u,x) + \deg a_{hl}(u,x) = \deg a_{il}(u,x) + \deg a_{hj}(u,x) \ \text{ for all } \ i,j,h,l.$$

Since

$$\deg(a_{i1}(u,x)) + h_{01} = \cdots = \deg(a_{is_0}(u,x)) + h_{0s_0} = h_{1i},$$

the matrix $A_\alpha = (a_{ij}(\alpha,x))$ is again a homogeneous matrix with

$$\deg(a_{i1}(\alpha,x)) + h_{01} = \cdots = \deg(a_{is_0}(\alpha,x)) + h_{0s_0} = h_{1i}.$$

Therefore, the homomorphism $\phi_\alpha : \bigoplus_{j=1}^{s_1} R_\alpha(-h_{1j}) \longrightarrow \bigoplus_{j=1}^{s_0} R_\alpha(-h_{0j})$ given by the matrix A_α is a graded homomorphism of degree 0.

Let L be a finitely generated graded R-module. Suppose that

$$\mathbf{F}_\bullet : \ 0 \longrightarrow F_\ell \xrightarrow{\phi_\ell} F_{\ell-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \longrightarrow L \longrightarrow 0$$

is a minimal graded free resolution of L, where each free module F_i may be written in the form $\bigoplus_j R(-j)^{\beta_{ij}}$, and all graded homomorphisms have degree 0. The following lemmas are well known and are needed afterwards.

Lemma 1.2. [5] *Let $\mathbf{F}_\bullet$ be a minimal graded free resolution of L. Then the complex*

$$(\mathbf{F}_\bullet)_\alpha : \ 0 \longrightarrow (F_\ell)_\alpha \xrightarrow{(\phi_\ell)_\alpha} (F_{\ell-1})_\alpha \longrightarrow \cdots \longrightarrow (F_1)_\alpha \xrightarrow{(\phi_1)_\alpha} (F_0)_\alpha \longrightarrow L_\alpha \longrightarrow 0$$

is a minimal graded free resolution of L_α with the same graded Betti numbers for almost all α.

Lemma 1.3. [5] *Let L be a finitely generated graded R-module. Then L_α is a graded R_α-module and $\dim_{k(\alpha)}(L_\alpha)_t = \dim_{k(u)} L_t, t \in \mathbb{Z}$, for almost all α.*

2. Irreducibility, Singularity of a hypersurface section

In this section we are interested in the intersection of a variety with a generic hypersurface. We will now begin by recalling the definition of Hilbert function.

Given any homogeneous ideal I of the standard grading polynomial ring $k[x] = k[x_0, \ldots, x_n]$ with $\deg x_i = 1$. We now set $R = k[x]/I = \bigoplus_{t \geqslant 0} R_t$. The Hilbert function of I, which is denoted by $h(-;I)$, is defined as follows $h(t;I) = \dim_k R_t$ for all $t \geqslant 0$. We make a number of simple observations, which are needed afterwards.

Lemma 2.1. *The Hilbert function is unchanged by projective inverse transformation. If k^* is an extension field of k, then $h(t;I) = h(t;Ik^*[x])$ for all $t \geqslant 0$.*

Lemma 2.2. *For two homogenous ideals I, J and a linear form ℓ of $k[x]$ with $I : \ell = I$ we have*
(i) $h(t;(I,J)) = h(t;I) + h(t;J) - h(t;I \cap J)$,
(ii) $h(t;(I,\ell)) = h(t;I) - h(t-1;I)$.

Proof. The equality (i) is obtained from the following exact sequence

$$0 \to k[x]/I \cap J \to k[x]/I \bigoplus k[x]/J \to k[x]/(I,J) \to 0,$$

where for $a, b \in k[x]$ the maps are $\overline{a} \to (\overline{a}, \overline{a})$ and $(\overline{a}, \overline{b}) \to \overline{a} - \overline{b}$. The equality (ii) is induced by (i).
For a set $X = \{q_i = (\eta_{i0}, \dots, \eta_{in}) \mid i = 1, \dots, s\}$ of s distinct K-rational points in $\mathbb{P}^n_K$, where K is an extension of k, we denote by $I = I(X)$ the homogeneous ideal of forms of $k[x]$ that vanish at all points of X. Let $k[x]/I$ be the homogeneous coordinate ring of X. The Hilbert function h_X of X is defined as follows

$$h_X(t) = h(t;I),\ \forall t \geqslant 0.$$

Before recalling the notion of groundform of an ideal we want to prove the Noetherian normalization of a homogeneous polynomial.

Lemma 2.3. *Assume that $t(x) \in k[x]$ is a homogeneous polynomial of degree s. There is a linear transformation and $a \in k$ such that $at(x)$ has the form*

$$at(x) = x_n^s + a_1(x)x_n^{s-1} + \cdots + a_s(x),$$

where $a_j(x) \in k[x_0, \dots, x_{n-1}]$ and $\deg a_j(x) \leqslant j$ or $a_j(x) = 0$.

Proof. We make a linear transformation $x_0 = y_0 + \lambda_0 y_n, \dots, x_{n-1} = y_{n-1} + \lambda_{n-1}y_n$ and $x_n = \lambda_n y_n$, where λ_i are undetermined constants of k. By this transformation, each power product of $t(x)$ is

$$\begin{aligned} x_0^{i_0} \dots x_{n-1}^{i_{n-1}} x_n^{i_n} &= (y_0 + \lambda_0 y_n)^{i_0} \dots (y_{n-1} + \lambda_{n-1}y_n)^{i_{n-1}} (\lambda_n y_n)^{i_n} \\ &= \lambda_0^{i_0} \dots \lambda_n^{i_n} y_n^s + \cdots . \end{aligned}$$

Denote $t(y_0 + \lambda_0 y_n, \dots, y_{n-1} + \lambda_{n-1}y_n, \lambda_n y_n)$ by $t(y)$. Then we can write

$$t(y) = b_0(\lambda)y_n^s + b_1(\lambda, y)y_n^{s-1} + \cdots + b_s(\lambda, y),$$

where $b_0(\lambda)$ is a nonzero polynomial in λ, and $b_j(\lambda, y) \in k[y_0, \dots, y_{n-1}]$. Since k is an infinite field, we can always choose $\lambda = (\lambda_0, \dots, \lambda_n) \in k^{n+1}$ such that $b_0(\lambda) \neq 0$. So for such a chosen λ, we write

$$\frac{1}{b_0(\lambda)} t(y) = y_n^s + a_1(\lambda, y)y_n^{s-1} + \cdots + a_s(\lambda, y).$$

By transformation $x_i = y_i, i = 0, \dots, n$, and chose $a = \frac{1}{b_0(\lambda)}$, the form $at(x)$ is what we wanted.

We proceed now to recall the notion of a ground-form which is introduced in order to study the properties of points on a variety. We consider an unmixed d-dimensional homogeneous ideal $P \subset k[x]$. Denote by $(v) = (v_{ij})$ a system of $(n+1)^2$ new indeterminates v_{ij}. We enlarge k by adjoining (v). The polynomial ring in $y_0, \dots, y_n$ over $k(v)$ will be denoted by $k(v)[y]$. The general linear transformation establishes an isomorphism between two polynomials rings $k(v)[x]$ and $k(v)[y]$ when in every polynomial of $k(v)[y]$ the substitution

$$y_i = \sum_{j=0}^{n} v_{ij} x_j,\ i = 0, 1, \dots, n,$$

is carried out. The inverse transformation

$$x_i = \sum_{j=0}^{n} w_{ij} y_j,\ i = 0, 1, \dots, n,$$

has its coefficients $w_{ij} \in k(v)$. We get $k(v)[x] = k(v)[y]$. Every ideal P of $k[x]$ generates an ideal $Pk(v)[x]$, which is transformed by the above isomorphism into the ideal

$$P^* = \big(\{f(\sum_{j=0}^{n} w_{0j} y_j, \sum_{j=0}^{n} w_{1j} y_j, \dots, \sum_{j=0}^{n} w_{nj} y_j) \mid f(x_0, x_1, \dots, x_n) \in P\}\big).$$

Then, the homogeneous ideal P in $k[x]$ transforms into the homogeneous ideal P^*, and the following ideal

$$P^* \cap k(v)[y_0, \dots, y_{d+1}] = (f(y_0, \dots, y_{d+1}))$$

with $\deg f(y_0, \dots, y_{d+1}) = s$ is clearly a principal ideal of $k(v)[y_0, \dots, y_{d+1}]$. By Lemma 2.3 we may suppose $f(y_0, \dots, y_{d+1})$ normalized so as to be a polynomial in the v_{ij}, and primitive in them, so that $f(y_0, \dots, y_{d+1})$ is defined to within a factor in $k(u, v)$. By a linear projective transformation, we can choose $f(y_0, \dots, y_{d+1})$ so that it is regular in y_{d+1}. The form $f(y_0, \dots, y_{d+1})$ is called a *ground-form* of P. If P is prime, then its ground-form is an irreducible form, but P is primary if and only if its ground-form is a power of an irreducible form. We emphasize that if P_1 and P_2 are distinct d-dimensional prime ideals, then the ground-form of P_1 is not a constant multiple of the ground-form of P_2, and the ground-form of a d-dimensional ideal is product of ground-forms of d-dimensional primary componentes, see [3, Satz 3 and Satz 4]. The concept of ground-form was formulated by E. Noether, see [3], [6]. More recent and simplified accounts can be found in W. Krull [3]. P^* has a monoidal prime basis

$$P^* = (f(y_0, \dots, y_{d+1}), a(y)y_{d+2} - a_2(y), \dots, a(y)y_n - a_n(y)),$$

where $a(y) \in k[y_0, \dots, y_d], a_i(y) \in k[y_0, \dots, y_{d+1}]$. Now the intersection of a variety with a hypersurface is interested.

Let $M_0, \dots, M_m$ be a fixed ordering of the set of monomials in $x_0, \dots, x_n$ of degree d, where $m = \binom{n+d}{n} - 1$. Let K be an extension of k. Giving a hypersurface f of degree d is the same thing as choosing $\alpha_0, \dots, \alpha_m \in K$, not all zero, and letting

$$f_\alpha = \alpha_0 M_0 + \cdots + \alpha_m M_m.$$

In other words, each hypersurface f_α of degree d can be presented as follows

$$f_\alpha = \alpha_0 x_0^d + \alpha_1 x_0^{d-1} x_1 + \cdots + \alpha_m x_n^d.$$

Let $u_0, \dots, u_m$ be the new indeterminates. The form $f_u = u_0 M_0 + \cdots + u_m M_m$ is called a *generic form* and $H_u = V(f_u)$ is called the *generic hypersurface*.

Theorem 2.4. *Let $V \subset \mathbb{P}_k^n$, $n \geqslant 3$, be a variety of dimension d, and let $H_\alpha = V(f_\alpha)$ be a hypersurface of $\mathbb{P}_{k(\alpha)}^n$ such that $V \not\subset V(f_\alpha)$ and $V \cap V(f_\alpha) \neq \emptyset$. Then the section $V \cap H_\alpha$ is again a variety of dimension $d-1$ for almost all α.*

Proof. Put $\mathfrak{p} = I(V)$. Suppose that $f_u = u_0 M_0 + \cdots + u_m M_m$ is the generic form. Since the irreducibility of a variety is preserved by finite pure transcendental extension of ground-field, V is still a variety in $\mathbb{P}_{k(u)}^n$. We have $I(V \cap H_u) = (\mathfrak{p}, f_u)$, and by [8, 34 Satz 2], the intersection $V \cap H_u$ is a variety of dimension $d-1$. Using a general linear transformation, the ground-form of $(\mathfrak{p}, f_u)$ can be assumed as a form $E(x_0, \dots, x_{d-1}, u, v)$. By [6, Theorem 6], $E(x_0, \dots, x_{d-1}, \alpha, v)$ is the ground-form of $(\mathfrak{p}, f_\alpha)$ or of $V \cap H_\alpha$. Since $V \cap H_u$ is a variety, $E(x_0, \dots, x_{d-1}, u, v)$ is a power of an irreducible form. Since $E(x_0, \dots, x_{d-1}, \alpha, v)$ is the same power of an irreducible form by [6, Lemma 8], $V \cap H_\alpha$ is again a variety. Because $\dim(\mathfrak{p}, f_\alpha) = \dim(\mathfrak{p}, f_u)$ by Lemma 1.1, $V \cap H_\alpha$ has the dimension $d-1$.

A variety V of $\mathbb{P}_k^n$ is *nondegenerate* if it does not lie in any hyperplane. Put $I(V) = \bigoplus_{j \geqslant 1} I_j$. Notice that V is nondegenerate if and only if $I_1 = 0$ or $h_V(1) = n+1$. We now consider the intersection $W = V \cap H$ of a nondegenerate variety V with a hyperplane

$$H : \ell = \alpha_0 x_0 + \cdots + \alpha_n x_n = 0.$$

From the above theorem it follows the following corollary.

Corollary 2.5. *Let V be a nondegenerate variety of $\mathbb{P}_k^n$ with $\dim V \geqslant 1$. Let $W = V \cap H_\alpha \subset H_\alpha \cong \mathbb{P}_{k(\alpha)}^{n-1}$ be a hyperplane section of V. Then W is again a nondegenerate variety of $\mathbb{P}_{k(\alpha)}^{n-1}$ with $\dim W = \dim V - 1$ if $\dim V > 1$ for almost all α. In the case $\dim V = 1$, W is a set of $s = \deg(V)$ points conjugate relative to $k(\alpha)$.*

Proof. By Theorem 2.4, W is a variety of dimension $\dim V - 1$. Set $\mathfrak{p} = I(V)$ and $\ell_u = u_0 x_0 + \cdots + u_n x_n$. Since $\mathfrak{p} k(u)[x] : \ell_u = \mathfrak{p} k(u)[x]$, by Lemma 2.1 and Lemma 2.2, we obtain

$$h(1; (\mathfrak{p}, \ell_u)) = h(1; \mathfrak{p}) - h(0; \mathfrak{p}) = n + 1 - 1 = n.$$

By Lemma 1.3, we have

$$h_W(1) = h(1; (\mathfrak{p}, \ell_\alpha)) = h(1; \mathfrak{p}) - h(0; \mathfrak{p}) = n + 1 - 1 = n.$$

Then $h_W(1) = n$. Hence W is again a nondegenerate variety of $\mathbb{P}^{n-1}_{k(\alpha)}$. In the case $\dim V = 1$, we get $\dim W = 0$. By Lemma 2.2, $\deg(W) = \deg(V)$, and therefore W is a set of $s = \deg(V)$ points conjugate relative to $k(\alpha)$.

3. Uniform position of a hyperplane section

Before coming to apply Harris' result about the set of points in uniform position we first shall need to recall here some definitions of points in $\mathbb{P}^n_k$. A set of s points, $X = \{q_1, \dots, q_s\}$ of $\mathbb{P}^n_k$, is said to be in *uniform position* if any two subsets of X with the same cardinality have the same Hilbert function. A The lemma of Harris [2] about a set of points in uniform position is the following

Lemma 3.1. [Harris's Lemma] *Let $V \subset \mathbb{P}^n_k$, $n \geqslant 3$, be an irreducible nondegenerate curve of degree s, and let H_u be a generic hyperplane of $\mathbb{P}^n_{k(u)}$. Then the section $V \cap H_u$ consists of s points in uniform position in $\mathbb{P}^{n-1}_{\overline{k(u)}}$.*

Upon simple computation, by repetition of Lemma 3.1 we obtain

Corollary 3.2. *Let $V \subset \mathbb{P}^n_k$, $n \geqslant 3$, be an irreducible nondegenerate variety of dimension $d > 0$ and of degree s, and let L_u be a generic linear subspace of dimension $h = n - d$ of $\mathbb{P}^n_{k(u)}$. Then the section $V \cap L_u$ consists of s points in uniform position in $\mathbb{P}^h_{\overline{k(u)}}$.*

Theorem 3.3. *Let $V \subset \mathbb{P}^n_k$, $n \geqslant 3$, be an irreducible nondegenerate variety of dimension $d > 0$ and of degree s, and let L_α be a linear subspace of dimension $h = n - d$ of $\mathbb{P}^n_k$ determined by linear forms*

$$f_i = \alpha_{i0}x_0 + \alpha_{i1}x_1 + \cdots + \alpha_{in}x_n, \ i = 1, \dots, d,$$

where $(\alpha) = (\alpha_{ij}) \in k^{d(n+1)}$. Then the section $V \cap L_\alpha$ consists of s points in uniform position for almost all α.

Proof. By L_u we denote a generic linear subspace of dimension $h = n - d$ of $\mathbb{P}^n_{k(u)}$ with defining equations

$$\ell_i = u_{i0}x_0 + u_{i1}x_1 + \cdots + u_{in}x_n, \ i = 1, \dots, d,$$

where $(u) = (u_{ij})$ is a family of $d(n+1)$ indeterminates u_{ij}. By Corollary 3.2, the section $V \cap L_u$ consists of s points in uniform position in $\mathbb{P}^h_{k(u)}$. The ideal

$$P = (I(V)k(u)[y], \ell_1, \dots, \ell_d)$$

is a 0-dimensional homogeneous prime ideal. We enlarge $k(u)$ by adjoining (v) and introduce the linear projective transformation

$$y_i = \sum_{j=0}^{n} v_{ij}x_j, \ i = 0, 1, \dots, n.$$

We get $k(u,v)[x] = k(u,v)[y]$, and the ideal P^* may be presented as

$$P^* = (f(u,v,y_0,y_1), a(u,v,y_0)y_2 - a_2(u,v,y_0,y_1), \dots, a(u,v,y_0)y_n - a_n(u,v,y_0,y_1)).$$

The form $f(u,v,y_0,y_1)$ is the ground-form of P. By substitution $(u,v) \to (\alpha)$ we obtain a linear subspace L_α of dimension $h = n-d$ of $\mathbb{P}^n_k$, by Lemma 1.1, determined by linear forms

$$(\ell_i)_\alpha = \alpha_{i0}x_0 + \alpha_{i1}x_1 + \cdots + \alpha_{in}x_n,\ i = 1,\dots,d.$$

The ideal of the section $V \cap L_\alpha$ is $P_\alpha = (I(V), (\ell_1)_\alpha, \dots, (\ell_d)_\alpha))$. Then

$$P^*_\alpha = (f(\alpha,y_0,y_1), a(\alpha,y_0)y_2 - a_2(\alpha,y_0,y_1), \dots, a(\alpha,y_0)y_n - a_n(\alpha,y_0,y_1)).$$

By [7, Theorem 6], the form $f(\alpha,y_0,y_1)$ is the ground-form of P_α. It is a specialization of $f(u,v,y_0,y_1)$. Since $V \cap L_u$ is irreducible, $f(v,y_0,y_1)$ is separable. It is well-known that $f(\alpha,y_0,y_1)$ is separable, too. There is

$$f(\alpha,y_0,y_1) = (y_1 - (\gamma_1)_\alpha y_0)\dots(y_1 - (\gamma_s)_\alpha y_0).$$

The zeros of $f(\alpha,1,y_1)$ are the specialization of zeros of $f(u,v,1,y_1)$. By Lemma 1.3, the proof is completed.

The set $Y = \{P_1,\dots,P_r\}$ is said to be *in generic position* if the Hilbert function satisfies $h_Y(t) = \min\{r, \binom{t+n}{n}\}$. The following result shows that almost all the section of an irreducible nondegenerate variety of dimension $d > 0$ and a linear subspace of dimension $h = n-d$ is a set of points in generic position

Corollary 3.4. *Let $V \subset \mathbb{P}^n_k$, $n \geqslant 3$, be an irreducible nondegenerate variety of dimension $d > 0$ and of degree s, and let L_α be a linear subspace of dimension $h = n-d$ of $\mathbb{P}^n_k$ determined by linear forms*

$$f_i = \alpha_{i0}x_0 + \alpha_{i1}x_1 + \cdots + \alpha_{in}x_n,\ i = 1,\dots,d,$$

where $(\alpha) = (\alpha_{ij}) \in k^{d(n+1)}$. Then the Hilbert function of every subset Y of the section $X = V \cap L_\alpha$ consisting r points, $r \in \{1,\dots,s\}$, satisfies $h_Y(t) = \min\{r, h_X(t)\}$ for almost all α.

Proof. By [1, Proposition 1.14], for any $r \in \{1,\dots,s\}$ there is a subcheme Z of of X consisting of r points such that $h_Z(t) = \min\{r, h_X(t)\}$. By Theorem 3.3, the Hilbert function of every subset Y of X consisting r points satisfies $h_Y(t) = h_Z(t)$ for almost all α. Hence $h_Y(t) = \min\{r, h_X(t)\}$ for almost all α.

Recall that a set of s points in $\mathbb{P}^n$ is called a *Cayley-Bachbarach scheme* if every subset of $s-1$ points has the same Hilbert function. As a sequence of Theorem 3.3 we have still the following corollary.

Corollary 3.5. *Let $V \subset \mathbb{P}_k^n$, $n \geqslant 3$, be an irreducible nondegenerate variety of dimension $d > 0$ and of degree s, and let L_α be a linear subspace of dimension $h = n - d$ of $\mathbb{P}_k^n$ determined by linear forms*

$$f_i = \alpha_{i0}x_0 + \alpha_{i1}x_1 + \cdots + \alpha_{in}x_n,\ i = 1, \ldots, d,$$

where $(\alpha) = (\alpha_{ij}) \in k^{d(n+1)}$. Then the section $X = V \cap L_\alpha$ is a Cayley-Bachbarach scheme for almost all α.

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