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On stability of Lyapunov exponents

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Abstract. In this paper we consider the upper (lower) - stability of Lyapunov exponents of linear differential equations in R^n. Sufficient conditions for the upper - stability of maximal exponent of linear systems under linear perturbations are given. The obtained results are extended to the system with nonlinear perturbations.
Keywork: Lyapunov exponents, upper (lower) - stability, maximal exponent.

1. Introduction

Let us consider a linear system of differential equations

$$\dot{x} = A(t)x; \;\; t \geq t_0 \geq 0. \tag{1}$$

where $A(t)$ is a real $n \times n$ - matrix function, continuous and bounded on $[t_0; +\infty)$. It is well known that the above assumption guarantees the boundesness of the Lyapunov exponents of system (1). Denote by

$$\lambda_1; \lambda_2; ...; \lambda_n \;\; (\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n)$$

the Lyapunov exponents of system (1).

Definition 1. *The maximal exponent λ_n of system (1) is said to be upper - stable if for any given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for any continuous on $[t_0; +\infty)$ $n \times n$ - matrix $B(t)$, satisfying $\|B(t)\| < \delta$, the maximal exponent μ_n of perturbed system*

$$\dot{x} = [A(t) + B(t)]x, \tag{2}$$

satisfies the inequality

$$\mu_n < \lambda_n + \epsilon. \tag{3}$$

If $\|B(t)\| < \delta$ implies $\mu_1 > \lambda_1 - \epsilon$, we say that the minimal exponent λ_1 of system (1) is lower - stable.

In general, the maximal (minimal) exponent of system (1) is not always upper (lower) - stable [1]. However, if system (1) is redusible (in the Lyapunov sense) then its maximal (minimal) exponent is upper (lower) - stable. In particular, if system (1) is periodic then it has this property [2,3]. A problem arises: In what conditions the maximal (minimal) exponent of nonreducible systems is upper (lower) - stable? The aim of this paper is to show a class of nonreducible systems, having this property.

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2. Preliminary lemmas

Lemma 1. *Let system* (1) *be regular in the Lyapunov sense. The maximal exponent* λ_n *is upper - stable if only if the minimal exponent of the adjoint system to* (1) *is lower - stable.*
Proof. We denote by

$$\alpha_1; \alpha_2; ...; \alpha_n \;\; (\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n)$$

the Lyapunov exponents of the adjoint system to (1):

$$\dot{y} = -A^*(t)y. \tag{4}$$

According to the Perron theorem, we have

$$\lambda_1 + \alpha_1 = 0, \quad \lambda_n + \alpha_n = 0. \tag{5}$$

If the maximal exponent λ_n of system (1) is upper - stable then the minimal exponent α_n of system (4) is lower - stable. In fact, denoting by

$$\beta_1; \beta_2; ...; \beta_n \;\; (\beta_1 \geq \beta_2 \geq ... \geq \beta_n)$$

the Lyapunov exponents of adjoint system to (2), we have

$$\beta_1 + \mu_1 = 0, \;\; \beta_n + \mu_n = 0. \tag{6}$$

Hence

$$\beta_n = -\mu_n > -\lambda_n - \epsilon = \alpha_n - \epsilon \;\; \text{if} \;\; \|B^*(t)\| < \delta. \tag{7}$$

Conversely, suppose that the minimal exponent α_n is lower - stable, then if (7) is satisfied we have

$$\beta_n \geq \alpha_n - \epsilon.$$

Then

$$\mu_n = -\beta_n < -\alpha_n + \epsilon = \lambda_n + \epsilon.$$

Which proves the lemma.

Consider now a nonlinear system of the form

$$\dot{x} = A(t)x + f(t, x). \tag{8}$$

Lemma 2. (Principle of linear inclusion) [1] *Let* $x(t)$ *be an any nontrivaial solution of system (8). There exists a matrix* $F(t)$ *such that* $x(t)$ *is a solution of the linear system*

$$\dot{y} = [A(t) + F(t)]y.$$

Moreover, if $f(t, x)$ *satisfies the condition*

$$\|f(t, x)\| \leq g(t)\|x\|; \; \forall t \geq t_0; \; \forall x \in R^n,$$

then matrix $F(t)$ *satisfies the inequality*

$$\|F(t)\| \leq g(t); \; \forall t \geq t_0.$$

The proof of Lemma 2 is given in [1].

3. Main results

3.1. Stability of system with the linear perturbations

In this section we consider systems of two linear differential equations in R^2:

$$\dot{x} = A(t)x \tag{9}$$

$$\dot{x} = A(t)x + B(t)x. \tag{10}$$

We denote by $\mu_1; \mu_2$ and $\lambda_1; \lambda_2$ ($\mu_1 \leq \mu_2;\ \ \lambda_1 \leq \lambda_2$) the exponents of systems (9) and (10) respectively. Let:

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}; \quad B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix}$$

We suppose that $A(t), B(t)$ are real matrix functions, continuous on $[t_0; +\infty)$ and $\sup_{t \geq t_0} \|A(t)\| = M < +\infty$.

Theorem 1. *Let system (9) be regular and there exists a constant $C > 0$ such that*

$$\int_{t_0}^{\infty} \sqrt{[a_{22}(t) - a_{11}(t)]^2 + [a_{21}(t) + a_{12}(t)]}^2 \ dt \leq C < +\infty,$$

then the maximal exponent λ_2 of system (9) *is upper - stable.*

Proof. Let

$$W(t) = \sqrt{[a_{22}(t) - a_{11}(t)]^2 + [a_{21}(t) + a_{12}(t)]}^2.$$

According to the Perron theorem [1,4] there exists an orthogonal matrix function $U(t)$ (i.e. $U^*(t) = U^{-1}(t),\ \forall t \geq t_0$) such that by the following transformation

$$x = U(t)y \tag{11}$$

the system $\dot{x} = A(t)x$ is reduced to

$$\dot{y} = P(t)y \tag{12}$$

where $P(t)$ is a matrix of the triangle form:

$$P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ 0 & p_{22}(t) \end{pmatrix}.$$

The matrix $P(t)$ is defined as $P(t) = U^{-1}(t)A(t)U(t) - U^{-1}(t)\dot{U}(t)$.

Now we show that if matrix $A(t)$ is bounded on $[t_0; +\infty)$, then matrix $P(t)$ is also bounded on this interval, i. e. exists a constant $M_1 > 0$ such that $\|P(t)\| \leq M_1$, $\forall t \geq t_0$. Indeed, let:

$$\tilde{A}(t) = (\tilde{a}_{ij}(t)) = U^{-1}(t)A(t)U(t);\ \ V(t) = (v_{ij}(t)) = U^{-1}(t)\dot{U}(t).$$

It is easy to show that $V^*(t) = -V(t)$. This implies $v_{ii}(t) = 0$, $\forall i = 1, 2$. Thus, we get

$$v_{ij}(t) = \begin{cases} -\tilde{a}_{ji}(t) & \text{if } \ i < j \\ 0 & \text{if } \ i = j \\ \tilde{a}_{ij}(t) & \text{if } \ i > j. \end{cases}$$

Since $A(t)$, $U(t)$, $U^{-1}(t)$ are bounded, matrix $P(t)$ is also bounded on $[t_0; +\infty)$. Let $\|P(t)\| \leq M_1, \forall t \geq t_0$. Taking the same Perron transformation to system (10), we obtain

$$\dot{x} = \dot{U}(t)y + U(t)\dot{y} = A(t)x + B(t)x$$

$$\Leftrightarrow U(t)\dot{y} = A(t)x + B(t)x - \dot{U}(t)y$$
$$\Leftrightarrow U(t)\dot{y} = A(t)U(t)y + B(t)U(t)y - \dot{U}(t)y$$
$$\Leftrightarrow \dot{y} = [U^{-1}(t)A(t)U(t) - U^{-1}(t)\dot{U}(t)]y + U^{-1}(t)B(t)U(t)y.$$

Denoting $Q(t) = U^{-1}(t)B(t)U(t)$, the last equation is in the form

$$\dot{y} = P(t)y + Q(t)y. \tag{13}$$

Writing triangle matrix $P(t)$ as follows:

$$P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ 0 & p_{22}(t) \end{pmatrix} = \begin{pmatrix} p_{11}(t) & 0 \\ 0 & p_{22}(t) \end{pmatrix} + \begin{pmatrix} 0 & p_{12}(t) \\ 0 & 0 \end{pmatrix}$$

and putting $\tilde{P}(t) = \begin{pmatrix} p_{11}(t) & 0 \\ 0 & p_{22}(t) \end{pmatrix}$; $\tilde{Q}(t) = Q(t) + \begin{pmatrix} 0 & p_{12}(t) \\ 0 & 0 \end{pmatrix}$,

we have

$$\dot{y} = \tilde{P}(t)y + \tilde{Q}(t)y. \tag{14}$$

Taking the linear transformation $y = Sz$ with

$$S = \begin{pmatrix} \frac{M_1}{\delta} & 0 \\ 0 & \sqrt{\frac{M_1}{\delta}} \end{pmatrix},$$

from (14) we get the following equivalent equation

$$\dot{z} = S^{-1}\tilde{P}(t)Sz + S^{-1}\tilde{Q}(t)Sz = \tilde{P}(t)z + S^{-1}\tilde{Q}(t)Sz. \tag{15}$$

Denoting by $\hat{Q}(\tau)$ the similar matrix of matrix $\tilde{Q}(\tau)$, we have

$$\hat{Q}(\tau) = S^{-1}\tilde{Q}(\tau)S = S^{-1}Q(\tau)S + S^{-1}\begin{pmatrix} 0 & p_{12}(\tau) \\ 0 & 0 \end{pmatrix} S,$$

which gives

$$\|\hat{Q}(\tau)\| \le \|S^{-1}Q(\tau)S\| + \|S^{-1}\begin{pmatrix} 0 & p_{12}(\tau) \\ 0 & 0 \end{pmatrix} S\|. \tag{16}$$

The solutions of the homogeneous system $\dot{z} = \tilde{P}(t)z$ is defined as follows

$$\dot{z} = \tilde{P}(t)z \Leftrightarrow \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} p_{11}(t) & 0 \\ 0 & p_{22}(t) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \Leftrightarrow \begin{cases} z_1(t) = C_1 e^{\int_{t_0}^{t} p_{11}(\tau)d\tau} \\ z_2(t) = C_2 e^{\int_{t_0}^{t} p_{22}(\tau)d\tau}. \end{cases}$$

Therefore

$$\Phi(t,\tau) = \begin{pmatrix} e^{\int_{t_0}^{t} p_{11}(s)ds - \int_{t_0}^{\tau} p_{11}(s)ds} & 0 \\ 0 & e^{\int_{t_0}^{t} p_{22}(s)ds - \int_{t_0}^{\tau} p_{22}(s)ds} \end{pmatrix}$$

is the Cauchy matrix of this system.

The solution satisfied the initial condition $z(t_0) = z_0$ of nonhomogeneous system (15) is given by [5]

$$z(t) = \Phi(t,t_0)z_0 + \int_{t_0}^{t} \Phi(t,\tau)S^{-1}\tilde{Q}(\tau)Sz(\tau)d\tau,$$

which is the same as $\Phi^{-1}(t,t_0)z(t) = z_0 + \int_{t_0}^{t} \Phi^{-1}(t,t_0)\Phi(t,\tau)S^{-1}\tilde{Q}(\tau)Sz(\tau)d\tau$

or $\Phi^{-1}(t,t_0)z(t) = z_0 + \int_{t_0}^{t} \Phi(t_0,\tau)S^{-1}\tilde{Q}(\tau)S\Phi(\tau,t_0)\Phi^{-1}(\tau,t_0)z(\tau)d\tau.$

Then

$$\|\Phi^{-1}(t,t_0)z(t)\| \le \|z_0\| + \int_{t_0}^{t} \|\Phi(t_0,\tau)S^{-1}\tilde{Q}(\tau)S\Phi(\tau,t_0)\|\|\Phi^{-1}(\tau,t_0)z(\tau)\|d\tau \tag{17}$$

$(t \ge \tau, s \ge t_0)$

Denoting by $\tilde{q}_{ij}(t)$ the elements of matrix $\tilde{Q}(t)$ and let

$$D = \Phi(t_0,\tau)S^{-1}\tilde{Q}(\tau)S\Phi(\tau,t_0),$$

we have

$$D = \begin{pmatrix} e^{-\int_{t_0}^{\tau} p_{11}(s)ds} & 0 \\ 0 & e^{-\int_{t_0}^{\tau} p_{22}(s)ds} \end{pmatrix} S^{-1} \begin{pmatrix} \tilde{q}_{11}(\tau) & \tilde{q}_{12}(\tau) \\ \tilde{q}_{21}(\tau) & \tilde{q}_{22}(\tau) \end{pmatrix} S \begin{pmatrix} e^{\int_{t_0}^{\tau} p_{11}(s)ds} & 0 \\ 0 & e^{\int_{t_0}^{\tau} p_{22}(s)ds} \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{q}_{11}(\tau) & \tilde{q}_{12}(\tau)e^{\int_{t_0}^{\tau}[p_{22}(s)-p_{11}(s)]ds} \\ \tilde{q}_{21}(\tau)e^{\int_{t_0}^{\tau}[p_{11}(s)-p_{22}(s)]ds} & \tilde{q}_{22}(\tau) \end{pmatrix}.$$

We can verify that

$$\left\| S^{-1} \begin{pmatrix} 0 & p_{12}(\tau) \\ 0 & 0 \end{pmatrix} S \right\| = \left\| \begin{pmatrix} 0 & p_{12}(\tau)\sqrt{\dfrac{\delta}{M_1}} \\ 0 & 0 \end{pmatrix} \right\| \le \sqrt{\delta}\sqrt{M_1}.$$

Since

$$\|Q(\tau)\| = \|U^{-1}(\tau)B(\tau)U(\tau)\| \le \|U^{-1}(\tau)\|\|B(\tau)\|\|U(\tau)\| \le 1.\delta.1 = \delta,$$

denoting $\max\{1+\sqrt{\frac{1}{M_1}}; 1+\sqrt{M_1}\} = M_2$ and chosing δ small enough such that $0 < \delta < 1$, we have

$$\|S^{-1}Q(\tau)S\| = \left\| \begin{pmatrix} q_{11}(\tau) & q_{12}(\tau)\sqrt{\frac{\delta}{M_1}} \\ q_{21}(\tau)\sqrt{\frac{M_1}{\delta}} & q_{22}(\tau) \end{pmatrix} \right\| \le \max\{\delta(1+\sqrt{\frac{\delta}{M_1}}); \delta(1+\sqrt{\frac{M_1}{\delta}}\}$$

$$= \max\{\sqrt{\delta}(\sqrt{\delta}+\delta\sqrt{\frac{1}{M_1}}; \sqrt{\delta}(\sqrt{\delta}+\sqrt{M_1}\} \le \sqrt{\delta}\max\{1+\sqrt{\frac{1}{M_1}}; 1+\sqrt{M_1}\} := \sqrt{\delta}M_2.$$

Consequently, applying the above inequalities to (16), we have $\|\hat{Q}(\tau)\| \le 2M_2\sqrt{\delta}$.

Now, we establish the norm of matrix D as follows:

It is known that in R^2 orthogonal matrix $U(t)$ has just one of two the following forms:

a) $U(t) = \begin{pmatrix} \cos\phi(t) & \sin\phi(t) \\ \sin\phi(t) & -\cos\phi(t) \end{pmatrix}$; b) $U(t) = \begin{pmatrix} \cos\phi(t) & -\sin\phi(t) \\ \sin\phi(t) & \cos\phi(t) \end{pmatrix}$.

Without loss of the generality we suppose that matrix $U(t)$ has the form a). In this case, we have

$$U^{-1}(t) = \begin{pmatrix} \cos\phi(t) & \sin\phi(t) \\ \sin\phi(t) & -\cos\phi(t) \end{pmatrix}.$$

Since in Perron transformation $x = U(t)y$, where $U(t)$ is a orthogonal matrix, the diagonal elements of matrix $P(t)$ and matrix $U^{-1}(t)A(t)U(t)$ are the same $p_{11}(t)$ and $p_{22}(t)$. Therefore we obtain that

$$p_{22}(t) - p_{11}(t) = [a_{22}(t)] - a_{11}(t)]\cos 2\phi(t) - [a_{21}(t) + a_{12}(t)]\sin 2\phi(t).$$

It is easy to see that, there is a function $\psi(t)$ such that

$$p_{22}(t) - p_{11}(t) = \sqrt{[a_{22}(t)] - a_{11}(t)]^2 + [a_{21}(t) + a_{12}(t)]^2}\ \cos[2\phi(t) + \psi(t)]$$

$$= W(t)\cos[2\phi(t) + \psi(t)].$$

Since $\|\tilde{q}_{ij}(t)\| \leq \|\tilde{Q}(t)\| \leq 2M_2\sqrt{\delta}$, we have

$$\|D\| = \left\| \begin{pmatrix} \tilde{q}_{11}(\tau) & \tilde{q}_{12}(\tau)e^{\int_{t_0}^{\tau}[p_{22}(s)-p_{11}(s)]ds} \\ \tilde{q}_{21}(\tau)e^{\int_{t_0}^{\tau}[p_{11}(s)-p_{22}(s)]ds} & \tilde{q}_{22}(\tau) \end{pmatrix} \right\|$$
$$\leq 2M_2\sqrt{\delta}[2 + e^{\int_{t_0}^{\tau}[p_{22}(s)-p_{11}(s)]ds} + e^{\int_{t_0}^{\tau}[p_{11}(s)-p_{22}(s)]ds}]$$
$$= 2M_2\sqrt{\delta}[2 + e^{\int_{t_0}^{\tau} W(s)\cos[2\phi(s)+\psi(s)]ds} + e^{\int_{t_0}^{\tau} W(s)\cos[2\phi(s)+\psi(s)-\pi]ds}].$$

From the assumptions $\int_{t_0}^{+\infty} W(t)dt \leq C < +\infty$, we have

$$\|D\| \leq 2M_2\sqrt{\delta}(2+2e^C) = M_3\sqrt{\delta} \text{ where } M_3 := 2M_2(2+2e^C).$$

Applying the last inequality to (17), we get

$$\|\Phi^{-1}(t,t_0)z(t)\| \leq \|z_0\| + \int_{t_0}^{t} M_3\sqrt{\delta}\|\Phi^{-1}(\tau,t_0)z(\tau)\|d\tau. \tag{18}$$

$(t \geq \tau, s \geq t_0)$
According to the Gronwall - Belman inequality [1, 4, 5], we have

$$\|\Phi^{-1}(t,t_0)z(t)\| \leq \|z_0\|e^{M_3\sqrt{\delta}\int_{t_0}^{t} d\tau} = \|z_0\|e^{M_3\sqrt{\delta}(t-t_0)}$$

$$\Rightarrow \begin{cases} e^{-\int_{t_0}^{t} p_{11}(\tau)d\tau} z_1(t) \leq \|z_0\|e^{M_3\sqrt{\delta}(t-t_0)} \\ e^{-\int_{t_0}^{t} p_{22}(\tau)d\tau} z_2(t) \leq \|z_0\|e^{M_3\sqrt{\delta}(t-t_0)} \end{cases} \Leftrightarrow \begin{cases} z_1(t) \leq \|z_0\|e^{M_3\sqrt{\delta}(t-t_0)}e^{\int_{t_0}^{t} p_{11}(\tau)d\tau} \\ z_2(t) \leq \|z_0\|e^{M_3\sqrt{\delta}(t-t_0)}e^{\int_{t_0}^{t} p_{22}(\tau)d\tau} \end{cases}.$$

Using properties of Lyapunov exponents, we get

$$\begin{cases} \chi[z_1] \leq \chi[\|z_0\|e^{M_3\sqrt{\delta}(t-t_0)}] + \chi[e^{\int_{t_0}^{t} p_{11}(\tau)d\tau}] = M_3\sqrt{\delta} + \lim_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t} p_{11}(\tau)d\tau \\ \chi[z_2] \leq \chi[\|z_0\|e^{M_3\sqrt{\delta}(t-t_0)}] + \chi[e^{\int_{t_0}^{t} p_{22}(\tau)d\tau}] = M_3\sqrt{\delta} + \lim_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t} p_{22}(\tau)d\tau. \end{cases}$$

It is clear that in Perron transformations the Lyapunov exponents are unchanged [1,4]. Thus, for any small enough given $\epsilon > 0$, chosing $0 < \delta < (\frac{\epsilon}{M_3})^2$, we obtain that

$$\begin{cases} \chi[x_1] = \chi[z_1] \leq \lambda_1 + \epsilon \\ \chi[x_2] = \chi[z_2] \leq \lambda_2 + \epsilon \end{cases} \quad \text{or} \quad \begin{cases} \mu_1 \leq \lambda_1 + \epsilon \\ \mu_2 \leq \lambda_2 + \epsilon. \end{cases}$$

The same result is proved for the case, when matrix $U(t)$ has form b).
The proof of theorem is completed.

Corollary 1. *Suppose that all assumptions of Theorem 1 hold. Then the minimal exponent of system* (9) *is lower - stable.*

Proof. From Lemma 1 it follows that minimal exponent of system (9) is lower - stable if the maximal exponent of adjoint system $\dot{x} = -A^*(t)x$ to this system is upper - stable. According to Theorem 1, the last requirement will be satisfied if the following inequality holds

$$\int_{t_0}^{\infty} \sqrt{[-a_{22}(t)+a_{11}(t)]^2 + [-a_{21}(t)-a_{12}(t)]^2}\ dt \leq C < +\infty$$
$$\Leftrightarrow \int_{t_0}^{\infty} \sqrt{[a_{22}(t)-a_{11}(t)]^2 + [a_{21}(t)+a_{12}(t)]^2}\ dt \leq C < +\infty.$$

This proves the corollary.

3.2. Stability of systems with nonlinear perturbations

We consider the following linear system with nonlinear perturbation in R^n:

$$\dot{x} = A(t)x + f(t, x). \tag{19}$$

Since the system (19) is nonlinear, it is dificult to study its spectrum [5]. However under the suitable conditions we can obtain some results on it, for example, to study supremum of its all exponents. Let us denote this supremum by μ_{sup}.

Definition 2. *The maximal exponent λ_n of homogeneous system $\dot{x} = A(t)x$ is said to be upper - stable under the nonlinear perturbation $f(t, x)$ if for any given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if following inequality holds $\|f(t, x)\| \leq \delta \|x\|$, then*

$$\mu_{\text{sup}} < \lambda_n + \epsilon. \tag{20}$$

We consider now the system (9) and (19) in R^2. For this space the following result is obtained:
Theorem 2. *Suppose that:*

i) System (9) is regular and there exists a constant $C > 0$ such that

$$\int_{t_0}^{\infty} \sqrt{[a_{22}(t) - a_{11}(t)]^2 + [a_{21}(t) + a_{12}(t)]}^2 \, dt \leq C < +\infty.$$

ii) Function $f(t, x)$ is continuous on $[t_0; +\infty)$ and there exists a function $g(t) > 0,\ \forall t \geq t_0$, satisfying the condition:

$$\|f(t, x)\| \leq g(t)\|x\|,\ \ \forall t \geq t_0$$

Then maximal exponent λ_2 of system (9) under perturbation $f(t, x)$ is upper - stable.
Proof. We denote by $x_0(t) = x(t_0, x_0, t)$ the solution of system (19), which satisfies initial condition $x_0(t_0) = x_0$. Denote by $F_{x_0}(t)$ the function matrix corresponding to this solution in the sense of Lemma 2, i.e. for this solution there exists a function matrix $F_{x_0}(t)$ such that $x_0(t)$ is a solution of the following linear system

$$\dot{x} = A(t)x + F_{x_0}(t)x,\ \ (x_0 \in R^2), \tag{21}$$

where $\|F_{x_0}(t)\| \leq g(t),\ \forall t \geq t_0$. We denote by $\mu_1^{x_0} \leq \mu_2^{x_0}$ the elements of spectrum of nonlinear system (19). According to Theomrem 1, for every given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|F_{x_0}(t)\| \leq \delta \quad \text{implies} \quad \mu_2^{x_0} < \lambda_2 + \frac{\epsilon}{2},\ \forall x_0 \in R^2.$$

From $\|F_{x_0}(t)\| \leq g(t) \leq \delta$, we have

$$\mu_2^{x_0} \leq \lambda_2 + \frac{\epsilon}{2},\ \forall x_0 \in R^2.$$

Therefore, we obtain that

$$\mu_{\text{sup}} = \sup_{x_0 \in R^2} \mu_2^{x_0} \leq \lambda_2 + \frac{\epsilon}{2} < \lambda_2 + \epsilon.$$

The proof is therefore completed.

Corollary 2. *Suppose that conditions $i)$ and $ii)$ of Theorem 2 hold and the function $g(t)$ in condition $ii)$ satisfies the condition*

$$\lim_{t\to+\infty} g(t) = 0.$$

Then maximal exponent λ_2 of system (9) under perturbation $f(t,x)$ is upper - stable.

Proof. For every given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|F_{x_0}(t)\| \le \delta \quad \text{implies} \quad \mu_2^{x_0} < \lambda_2 + \frac{\epsilon}{2},\ \forall x_0 \in R^2.$$

Since $\lim_{t\to+\infty} g(t) = 0$, for $\delta > 0$ there exists $T = T(\delta) \ge t_0$ such that $0 < g(t) < \delta,\ \forall t \ge T$. Thus, if $t \ge T$ then $\|F_{x_0}(t)\| \le g(t) \le \delta$. Taking to limit as $t \to +\infty$, we have

$$\mu_2^{x_0} \le \lambda_2 + \frac{\epsilon}{2},\ \forall x_0 \in R^2.$$

Taking to supremum over all $x_0 \in R^2$, we have

$$\mu_{\sup} = \sup_{x_0\in R^2} \mu_2^{x_0} \le \lambda_2 + \frac{\epsilon}{2} < \lambda_2 + \epsilon.$$

The proof is therefore completed.

Example. Consider the system

$$\begin{cases} \dot{x}_1 = (1 + \dfrac{1}{t^2})x_1 \\ \dot{x}_2 = \dfrac{\sqrt{3}}{t^2}x_1 + (1 + \dfrac{2}{t^2})x_2 \\ t \ge 1. \end{cases} \tag{22}$$

It is easy to see that this system is nonredusible and nonperiodic. We can show that for this system:

$$\lambda_1 = \lambda_2 = 1 \text{ and } \lim_{t\to+\infty} \frac{1}{t}\int_1^t SpA(s)ds = 2.$$

Therefore, system (22) is regular. We can see also for this system:

$$W(t) = \sqrt{[(1+\frac{2}{t^2}) - (1+\frac{1}{t^2})]^2 + (\frac{\sqrt{3}}{t^2})^2} = \frac{2}{t^2}.$$

Therefore, we get

$$\int_1^t W(s)ds = 2 - \frac{2}{t} \le 2,\ \forall t \ge 1.$$

Thus, system (22) satisfies all conditions of Theorem 1. Its maximal exponent is upper - stable.

References

[1] B.F. Bulop, P.E. Vinograd, D.M. Gropman, B.B. Nemeskii, *Theory of Lyapunov exponents and applications to stability problems,* Nauka, Moscow (1966) (in Russian).

[2] Nguyen The Hoan, Dao Thi Lien, On uniform stability of characteristic spectrum for sequences of linear equation systems, *Journal of Science, VNU* T XV (5) (1999) 28.

[3] Nguyen Minh Man, Nguyen The Hoan, On some asymptotic behaviour for solutions of linear differential equations, *Ucrainian Math. Journal* Vol 55 (4) (2003) 501.

[4] B.P. Demidovich, *Lectures on Mathematical Theory of Stability*, Nauka (1967) (in Russian).

[5] N.S. Bay, V.N. Phat, Asymptotic stability of a class of nonlinear functional differential equations, *Nonlinear Functional Analysis and Applications* Vol 7 (2) (2002) 299.