ON THE STABILITY OF ELASTOPLASTIC THIN TRIANGULAR PLATES MADE IN COMPRESSIBLE MATERIAL

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Abstract. The stability problem of thin triangular plates by the small elastoplastic deformation theory, was studied in [3]. Basing on the theory of elastoplastic processes, this problem again has been investigated in [4] with incompressible material.

In this paper we continue to study the mentioned problem with compressible material. The relation for determining critical forces is established. In particular the explicit expression of the critical force for the linear hardening material is found. Some numerical calculations have been given and discussed.

1. Problem setting and fundamental stability equations

Let's consider a isosceles right triangular thin plate with the right side a and thickness h. We choose a orthogonal coordinate system $Oxyz$ so that the axis x and y coincide with two right sides of plate, the axis z in direction of the normal to the middle surface.

Assume that a material is compressible and the plate is subjected to the compressible forces with the intensity uniformly distributed $p = p(t)$ at the sides $x = 0$, $y = 0$ and $x + y = a$, where t - loading parameter. Moreover we suppose don't take into account the unloading in the plate. The problem is to have to find the critical value $t = t_*$ and respectively the critical load $p_* = p(t_*)$ which at that time t_* an instability of the structure appears. We use the crirerion of bifurcation of equilibrium state to investigate the proposed problem.

1.1. Pre-buckling state

At any moment t in the plate, there exists the plane stress state

$$\begin{aligned} &\sigma_{xx} = -p(t) \equiv -p, \quad \sigma_{yy} = -p(t) \equiv -p, \\ &\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0. \end{aligned} \tag{1.1}$$

So that

$$\sigma = -\frac{2}{3}p; \quad \sigma_u = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy}} = p. \tag{1.2}$$

The material is assumed to be compressible, i.e $\sigma = 3K\varepsilon$. So $\varepsilon = \dfrac{\sigma}{3K} = -\dfrac{2p}{9K}$, $K = \dfrac{E}{3(1-2\nu)}$, $E = 2G(1+\nu)$, where K is compressible coefficient of material. The components

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX of the strain velocity tensor detemined by the stress-strain relationship of elastoplastic process theory [1, 7] are of the form

$$\dot{\varepsilon}_{xx} = \dot{\varepsilon}_{yy} = -\Big(\frac{1}{2\phi'} + \frac{2}{9K}\Big)\dot{p}, \quad \dot{\varepsilon}_{xy} = \dot{\varepsilon}_{yz} = \dot{\varepsilon}_{xz} = 0,$$
$$\dot{\varepsilon}_{zz} = \Big(\frac{1}{\phi'} - \frac{2}{9K}\Big)\dot{p}; \quad \phi' = \phi'(s), \tag{1.3}$$

where s is the arc-length of the strain trajectory calculated by the formula

$$\frac{ds}{dt} = \frac{\sqrt{2}}{3}\Big[(\dot{\varepsilon}_{xx} - \dot{\varepsilon}_{yy})^2 + (\dot{\varepsilon}_{yy} - \dot{\varepsilon}_{zz})^2 + (\dot{\varepsilon}_{zz} - \dot{\varepsilon}_{xx})^2\Big]^{1/2} = \frac{\dot{p}}{\phi'(s)}$$

or $\quad \phi'(s)ds = dp.$

That yields

$$p = \sigma_u = \phi(s), \tag{1.4}$$

or with the hardening material, $s = \phi^{-1}(p)$.

nn*1.2. Post-buckling state*

At the moment an instability occurs, a bifurcation of equilibrium states is assumed to appear. The system of stability equations of the compressible thin plates presented in [7] is written in form as follows

$$\alpha_1\frac{\partial^4\delta w}{\partial x^4} + \alpha_3\frac{\partial^4\delta w}{\partial x^2\partial y^2} + \alpha_5\frac{\partial^4\delta w}{\partial y^4} + \frac{9p}{h^2N}\Big(\frac{\partial^2\delta w}{\partial x^2} + \frac{\partial^2\delta w}{\partial y^2}\Big) = 0. \tag{1.5}$$

where

$$\alpha_1 = \alpha_5 = \frac{1}{C}\Big(\frac{1}{4} + \frac{3\phi'}{4N} + \frac{\phi'}{9K}\Big); \quad \alpha_3 = \frac{1}{C}\Big(\frac{1}{2} + \frac{3\phi'}{2N} - \frac{2\phi'}{9K}\Big),$$
$$N = \frac{\sigma_u}{s} = \frac{p}{s}, \quad C = 1 + \frac{4\phi'}{9K}, \quad \phi' = \phi'(s). \tag{1.6}$$

1.3. Boundary conditions

We consider the thin plate with the simply supported boundary conditions. In this case we have

$$\delta w = 0, \quad \frac{\partial^2\delta w}{\partial x^2} = 0, \qquad \text{at } x = 0,$$
$$\delta w = 0, \quad \frac{\partial^2\delta w}{\partial y^2} = 0, \qquad \text{at } y = 0,$$
$$\delta w = 0, \quad \frac{3\phi'}{C}(\Delta\delta w) + 2N\frac{\partial^2\delta w}{\partial x\partial y} = 0, \quad \text{at } x + y = a. \tag{1.7}$$

2. Solving method

We choose the deflection δw satisfing the boundary condition (1.7) in the form

$$\delta w = A_{mn}\Big[\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} + (-1)^{m+n+1}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}\Big],\ (m,n\in N^+; m\neq n) \quad (2.1)$$

Calculating partial derivatives of δw and substituting those expressions into the stability equation (1.5) and taking into account the existence of non-trivial solution i.e $A_{mn}\neq 0$, we receive the expression

$$\alpha_1\Big(\frac{m\pi}{a}\Big)^4 + \alpha_2\Big(\frac{m\pi}{a}\Big)^2\Big(\frac{n\pi}{a}\Big)^2 + \alpha_5\Big(\frac{n\pi}{a}\Big)^4 - \frac{9p}{Nh^2}(m^2+n^2)\frac{\pi^2}{a^2} = 0. \quad (2.2)$$

By putting $i = \dfrac{3a}{h}$ (called the slenderness of the plate) and $\alpha_1 = \alpha_5$, the relation (2.2) becomes

$$i^2 = \frac{9a^2}{h^2} = \frac{N\pi^2}{p}\,\frac{\alpha_1(m^2+n^2)^2 + (\alpha_3 - 2\alpha_1)m^2n^2}{m^2+n^2}\,. \quad (2.3)$$

Substituting the values of α_1 and α_3 into (2.3) we get

$$i^2 = \frac{9a^2}{h^2} = \frac{\pi^2}{s}\Big[\frac{1}{C}\Big(\frac{1}{4} + \frac{3\phi' s}{4p} + \frac{\phi'}{9K}\Big)(m^2+n^2) - \frac{4\phi' m^2n^2}{9KC(m^2+n^2)}\Big]. \quad (2.4)$$

This equation (2.4) permits us to determine a critical load p_*. Because of the force p is non-linear function of s, then the relation (2.4) is too non-linear to s.

We can solve this equation by using the modified elastic solution method [2].

First of all, choosing $m = 1$, $n = 2$, the equation (2.4) can be rewritten in the other form

$$s = \frac{5\pi^2}{4}\Big(\frac{h}{a}\Big)^2\Big(1 + \frac{3s\phi'}{\phi}\Big)\frac{K}{9K+4\phi'} + \frac{\pi^2}{5}\Big(\frac{h}{a}\Big)^2\cdot\frac{\phi'}{9K+4\phi'} \quad (2.5)$$

or

$$s = \frac{5\pi^2}{4}\Big(\frac{h}{a}\Big)^2\Big[1 + 3\frac{E_t(s)}{E_c(s)}\Big]\frac{K}{9K+4E_t(s)} + \frac{\pi^2}{5}\Big(\frac{h}{a}\Big)^2\frac{E_t(s)}{9K+4E_t(s)}\,, \quad (2.6)$$

where $E_t(s) = \phi'(s)$ - the tangential modulus, $E_c(s)$ - the secant modulus of the material. The problem determining critical loads of a plate reduces to seek the critical value s_*. Finally the critical load can be found from

$$p_* = \phi(s_*). \quad (2.7)$$

Now we presente in detail this iterative method.

On the first iteration by putting $E_c(s) = E_t(s) = 3G$, from (2.6) we get

$$s_1 = 5\pi^2\Big(\frac{h}{a}\Big)^2\frac{K}{9K+12G} + \frac{\pi^2}{5}\Big(\frac{h}{a}\Big)^2\frac{G}{3K+4G}\,. \quad (2.8)$$

If $s_1 \leqslant \varepsilon_s$ (elastic limit), the iteration is finished and the critical force is given by

$$p_*^{(1)} = 3Gs_1 = 5G\pi^2\Big(\frac{h}{a}\Big)^2\frac{K}{3K+4G} + \frac{3\pi^2}{5}\Big(\frac{h}{a}\Big)^2\frac{G^2}{3K+4G}\,. \quad (2.9)$$

If $s_1 > \varepsilon_s$, we proceed to the second iteration by the formula

$$s_2 = \frac{5\pi^2}{4}\Big(\frac{h}{a}\Big)^2\Big[1 + 3\frac{E_t(s_1)}{E_C(s_1)}\Big]\frac{K}{9K + 4E_t(s_1)} + \frac{\pi^2}{5}\Big(\frac{h}{a}\Big)^2\frac{E_t(s_1)}{9K + 4E_t(s_1)}\,. \tag{2.10}$$

The calculations are realized analogously as the first iteration.

A procedure of the iterative method for solving the relation (2.6) can be written as following

$$s_n = \frac{5\pi^2}{4}\Big(\frac{h}{a}\Big)^2\Big[1 + 3\frac{E_t(s_{n-1})}{E_c(s_{n-1})}\Big]\frac{K}{9K + 4E_t(s_{n-1})} + \frac{\pi^2}{5}\Big(\frac{h}{a}\Big)^2\frac{E_t(s_{n-1})}{9K + 4E_t(s_{n-1})} \tag{2.11}$$

and the critical force for n-th iteration, is determined by

$$p_*^{(n)} = \phi(s_n), \tag{2.12}$$

where s_{n-1} is considered to be known at $(n-1)$-th iteration.

Practically, the iterative process will be finished when

$$\Big|\frac{s_n - s_{n-1}}{s_{n-1}}\Big| < \varepsilon, \tag{2.13}$$

where ε is a given forward positive and small value.

3. Linear hardening material

The general case for hardening material is presented in the above part, now we consider the problem for linear hardening material.

3.1. If the function $\sigma_u = \phi(s)$ is represented by graph in figure 1.

In this case we have $\phi' = g = \text{const}$, $\sigma_u = p = 3Gs_0 + (s - s_0)g$. It is seen from here that

$$s = \frac{p - (3G - g)s_0}{g} \equiv \frac{p - \lambda}{g} \tag{3.1}$$

where $\lambda = (3G - g)s_0 = \Big(1 - \dfrac{g}{3G}\Big)\sigma_s$; $0 \leqslant \lambda \leqslant \sigma_s$, σ_s is an upper limit of elastic stress.

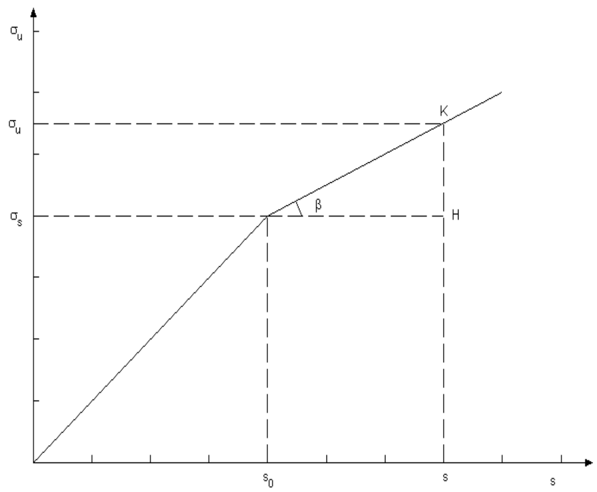

Figure 1

Substituting the expression of s from (3.1) into (2.4), we obtain the equation for finding the critical force p as follows

$$36a^2p^2 - \Big\{36\lambda a^2 + \frac{4g\pi^2h^2}{C^2}(m^2+n^2)\Big[1+\frac{g}{9K}\Big(\frac{m^2-n^2}{m^2+n^2}\Big)^2\Big]\Big\}p + \frac{3h^2\pi^2 g\lambda}{C}(m^2+n^2) = 0. \tag{3.2}$$

Putting the left side of (3.2) equal to $f(p)$, we notice that $f(p)$ is the continuous function to s and $f(\lambda) \leqslant 0$. So that the equation (3.2) gives us two solutions satisfying the conditions $p_1 \leqslant \lambda \leqslant p_2$. Solving the equation (3.2), finally we have

$$p = \frac{1}{18a^2}\Bigg\{9\lambda a^2 + \frac{g\pi^2h^2}{C}(m^2+n^2)\Big[1+\frac{g}{9K}\Big(\frac{m^2-n^2}{m^2+n^2}\Big)^2\Big] + \sqrt{\Big\{9\lambda a^2 + \frac{g\pi^2h^2}{C}(m^2+n^2)\Big[1+\frac{g}{9K}\Big(\frac{m^2-n^2}{m^2+n^2}\Big)^2\Big]\Big\}^2 - \frac{27\pi^2h^2g\lambda a^2}{C}(m^2+n^2)}\Bigg\}. \tag{3.3}$$

Remarks

+ If material is elastic i.e $g = 3G$, the expression (3.3) becomes

$$p = \frac{\pi^2 G}{3C}(m^2+n^2)\frac{h^2}{a^2}\Big[1+\frac{G}{3K}\Big(\frac{m^2-n^2}{m^2+n^2}\Big)^2\Big] \tag{3.4}$$

+ If material is incompressible i.e $K \to \infty$, the expression (3.3) is given

$$p = \frac{1}{18a^2}\Big\{9\lambda a^2 + gh^2\pi^2(m^2+n^2) + \sqrt{\big[9\lambda a^2 + gh^2\pi^2(m^2+n^2)\big]^2 - 27g\pi^2h^2\lambda a^2(m^2+n^2)}\Big\}. \tag{3.5}$$

Deduces from here

$$p_* = \min p = p\big|_{m^2+n^2=5} = \frac{1}{18a^2}\Big\{9\lambda a^2 + 5g\pi^2h^2 + \sqrt{(9\lambda a^2+5g\pi^2h^2)^2 - 135g\lambda a^2\pi^2h^2}\Big\}. \tag{3.6}$$

This result coincides with one presented in [4].

3.2. If the function $\sigma_u = \phi(s)$ is represented by graph in figure 2.

We have

$$\sigma_u = \sigma_s + (s_1 - s_0)\mathrm{tg}\alpha_1 + \cdots + (s_{k-1}-s_{k-2})\mathrm{tg}\alpha_{k-1} + (s - s_{k-1})\mathrm{tg}\alpha_k = 3Gs_0 + \sum_{i=1}^{k-1}(s_i - s_{i-1})g_i + (s-s_{k-1})g_k, \tag{3.7}$$

where $s_0 = \dfrac{\sigma_s}{3G}$; $g_i = \text{tg}\alpha_i = \phi'(s)$ with $s_{i-1} \leqslant s \leqslant s_i$; $i = \overline{1, k-1}$; $g_k = \text{tg}\alpha_k = \phi'(s)$ with $s \geqslant s_k$.

Because $\sigma_u = p$, so (3.7) leads

$$p = (3G - g_1)s_0 + \sum_{i=1}^{k-1}(g_i - g_{i+1})s_i + g_k s. \tag{3.8}$$

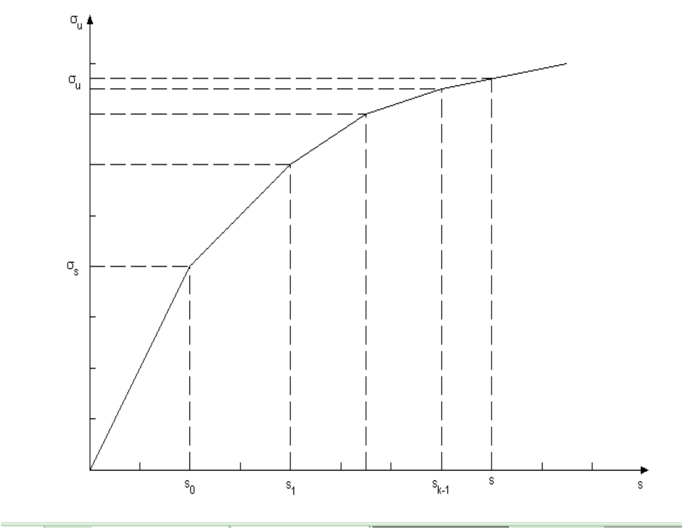

Figure 2

Deduces from here

$$s = \frac{p - \left[(3G - g_1)s_0 + \sum\limits_{i=1}^{k-1}(g_i - g_{i+1})s_i\right]}{g_k} \equiv \frac{p-\lambda}{g_k}, \tag{3.9}$$

where

$$\lambda = (3G - g_1)s_0 + \sum_{i=1}^{k-1}(g_i - g_{i+1})s_i.$$

Substituting (3.9) into (2.4) and calculating analogously as the part 3.1, we get

$$p = \frac{1}{18a^2}\left\{9\lambda a^2 + \frac{g_k\pi^2h^2}{C}(m^2+n^2)\left[1 + \frac{g_k}{9K}\left(\frac{m^2-n^2}{m^2+n^2}\right)^2\right] + \right. \tag{3.10}$$

$$\left. \sqrt{\left\{9\lambda a^2 + \frac{g_k\pi^2h^2}{C^2}(m^2+n^2)\left[1 + \frac{g_k}{9K}\left(\frac{m^2-n^2}{m^2+n^2}\right)^2\right]\right\}^2 - \frac{27\pi^2h^2g_k\lambda a^2}{C}(m^2+n^2)}\right\}.$$

This is the relation for determining the critical force p_*. It is seen that if $g_1 = g_2 = \cdots = g_k = g$, the expression (3.10) returns to the result (3.3).

4. Numerical calculations and discussion

4.1. Linear hardening material

We consider a plate with the characteristics as follows $3G = 2.6 \cdot 10^5$ (MPa), $\sigma_s = 400$ (MPa), $\phi'(s) = g = 0.208 \cdot 10^5$ (MPa), m, n from 1 to 10 ($m \neq n$). The ratio $\dfrac{a}{h}$ varies from 22 to 49 with the arithmetical ratio equal to 3, $K = \dfrac{E}{3(1-2\nu)}$, $E = 2G(1+\nu)$, ν from 0.20 to 0.50 with arithmetical ratio equal to 0.04. We use the formula (3.3).

Hereafter we give the numerical results which are represented by graphs

a) Plastic σ_u^* and elastic σ_u^* in the cases $\nu = 0.2$ (table 3, figure 3)

b) Plastic σ_u^* and elastic σ_u^* with $\nu = 0.44$ (table 4, figure 4)

c) Plastic σ_u^* and elastic σ_u^* with $\nu = 0.5$ (table 5, figure 5)

Table 3

a/h	$\sigma_u^*(plastic)(MPa)$	$\sigma_u^*(elastic)(MPa)$
22	455.571	2972.747
25	429.571	2303.095
28	413.315	1835.216
31	403.517	1497.200
34	396.405	1244.644
37	391.282	1050.993
40	387.462	899.255
43	384.531	778.156
46	382.229	679.967
49	380.386	599.254

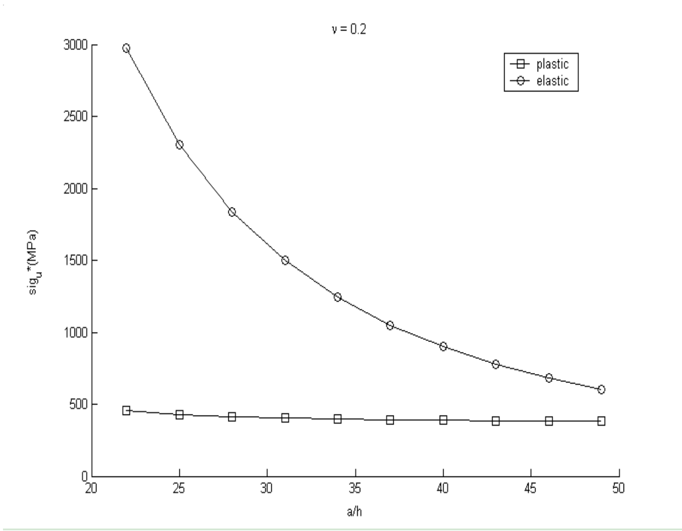

Figure 3

Table 4

a/h	$\sigma_u^*(plastic)(MPa)$	$\sigma_u^*(elastic)(MPa)$
22	461.854	2950.319
25	433.693	2284.473
28	416.628	1821.370
31	405.552	1485.905
34	397.952	1235.255
37	392.500	1043.063
40	388.447	892.471
43	385.345	772.285
46	382.915	674.837
49	380.972	594.733

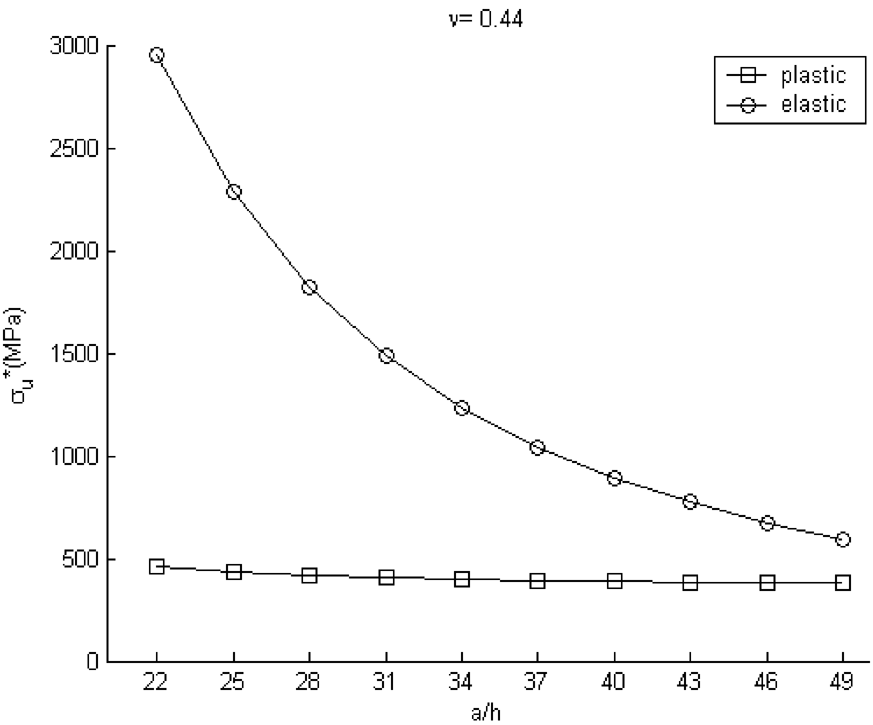

Figure 4

Table 5

a/h	$\sigma_u^*(plastic)(MPa)$	$\sigma_u^*(elastic)(MPa)$
22	436.246	2945.474
25	434.589	2280.975
28	417.244	1818.379
31	406.000	1483.456
34	398.292	1233.226
37	392.767	1041.351
40	388.662	891.006
43	385.532	771.016
46	383.064	673.728
49	381.099	593.756

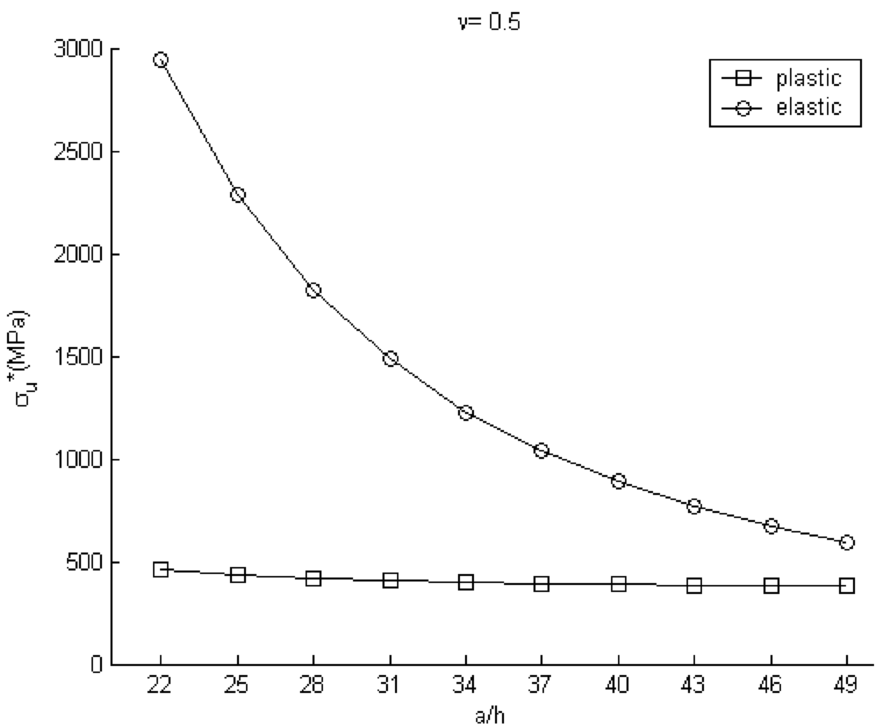

Figure 5

4.2. Hardening material

We consider a plate made of the stell 30XΓCA with an elastic modulus $3G = 2.6 \cdot 10^5$ MPa, an yield point $\sigma_u = 400$ MPa and the table de dates given in [1]. The Poisson coefficient is equals to 0.2; 0.32; 0.44. The calculations are realized by the formula

(2.6) and the iterative method represented in part 2. Finally we receive the results in the table 6 and the figure 6.

Table 6

a/h	$\sigma_u^*(\nu = 0.2)$	$\sigma_u^*(\nu = 0.32)$	$\sigma_u^*(\nu = 0.44)$
22	531.498	544.375	568.888
25	511.295	528.059	544.254
28	497.993	510.571	529.782
31	481.474	498.883	515.507
34	466.801	485.054	502.241
37	451.065	472.045	489.882
40	435.384	456.226	476.926
43	407.437	444.688	466.564
46	367.182	426.108	452.499
49	323.597	397.762	443.272

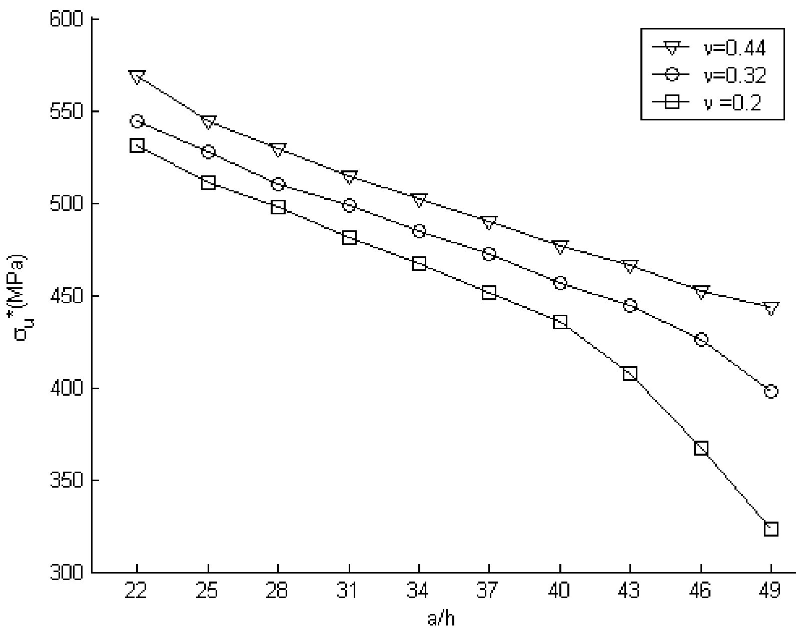

Figure 6

Discussion

The above received results lead us to some remarks as follows

a) The more the plate is thin, the more the value of critical stress intensity σ_u^* is small (see Figures 3, 4, 5, 6).

b) The compressibility of material has an influence on the stability of structure. The more the Poisson coefficient ν decreases, the more the value of σ_u^* diminishes when the ratio $\frac{a}{h}$ is constant. This remark is deduced from the results in Table 6 and Figure 6.

c) When a material is incompressible, the obtained results return to the previous well-known ones (see [3, 4, 5, 6, 8]).

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