VNU Journal of Science, Mathematics - Physics 23 (2007) 22-27

Eliminating on the divergences of the photon self - energy diagram in (2+1) dimensional quantum electrodynamics

Nguyen Suan Han[1], Nguyen Nhu Xuan[2,*]

[1]*Department of Physics, College of Science, VNU*
334 Nguyen Trai, Hanoi, Vietnam
[2]*Department of Physics, Le Qui Don Technical University*

Received 15 May 2007

Abstract: The divergence of the photon self-energy diagram in spinor quantum electrodynamics in $(2+1)$ dimensional space time- (QED_3) is studied by the Pauli-Villars regularization and dimensional regularization. Results obtained by two different methods are coincided if the gauge invariant of theory is considered carefully step by step in these calculations.

1. Introduction

It is well known that the gauge theories in $(2+1)$ dimensional space time though super-renormalizable theory [1], showing up inconsistence already at one loop, arising from the regularization procedures adopted to evaluate ultraviolet divergent amplitudes such as the photon self-energy in QED_3. In the latter, if we use dimensional regularization [2] the photon is induced a topological mass in contrast with the result obtained through the Pauli-Villars scheme [3], where the photon remains massless when we let the auxiliary mass go to infinity. Other side this problem is important for constructing quantum field theory with low dimensional modern.

This report is devoted to show up the inconsistencies not arising in QED_3, if the gauge invariance of theory is considered carefully step by step in those calculations by above methods of regularization for the photon self-energy diagram. The paper is organized as follows. In the second section the photon self-energy is calculated by the dimensional regularization. In the third section this problem is done by the Pauli-Villars method. Finally, we draw our conclusions.

2. Dimensional regularization

In this section, we calculate the photon self-energy diagram in QED_3 given by (Fig. 1)

* Corresponding author. Tel: 84-4-069515341
E-mail: xuan_76@yahoo.com

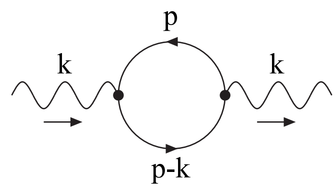

Figure 1. The photon self-energy diagram.

Following the standard notation, this graph is corresponding to the formula:

$$\Pi_{\mu\nu}(k) = \frac{ie^2}{(2\pi)^{\frac{3}{2}}} \int d^3pTr\left[\gamma_\mu \frac{\hat{p}+m}{p^2-m^2+i\epsilon}\gamma_\nu \frac{\hat{p}-k+m}{(p-k)^2-m^2+i\epsilon}\right]. \tag{1}$$

In dimensional regularization scheme, we have to make the change :

$$\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \to \mu^\epsilon \int \frac{d^np}{(2\pi)^{\frac{n}{2}}}, \tag{2}$$

where $\epsilon = 3-n, \mu$ is some arbitrary mass scale which is introduced to preserve dimensional of system.

Make to shift p by $p+\frac{1}{2}k$, the expression (1) has the form :

$$\begin{aligned}\Pi_{\mu\nu}(k) &= ie^2\mu^\epsilon \int \frac{d^np}{(2\pi)^{\frac{n}{2}}}\left[\gamma_\mu \frac{\left(\hat{p}+\frac{1}{2}\hat{k}\right)+m}{\left(p+\frac{1}{2}k\right)^2-m^2+i\epsilon}\gamma_\nu \frac{\left(\hat{p}-\frac{1}{2}\hat{k}\right)+m}{(p-\frac{1}{2}k)^2-m^2+i\epsilon}\right] \\ &= ie^2\mu^\epsilon \int \frac{d^np}{(2\pi)^{\frac{n}{2}}}\int_0^1 dx \frac{P(m)}{\left[m^2-p^2+(x^2-x)k^2\right]^2},\end{aligned} \tag{3}$$

with

$$\begin{aligned}P(m) =&2\Big\{m^2 g_{\mu\nu}+2p_\mu p_\nu+(1-2x)p_\nu k_\mu+2(x^2-x)k_\mu k_\nu \\ &- g_{\mu\nu}\left[p^2+(1-2x)pk+(x^2-x)k^2\right]-im\epsilon_{\mu\nu\alpha}k^\alpha\Big\}.\end{aligned} \tag{4}$$

In the expression (3), we have used Feynman integration parameter [5].

Neglecting the integrals that contain the odd terms of p in $P(m)$ which will vanish under the symmetric integration in p. Then we have

$$\begin{aligned}\Pi_{\mu\nu}(k) =&2ie^2\mu^\epsilon \int_0^1 dx \int \frac{d^np}{(2\pi)^{\frac{n}{2}}}\times \\ &\left\{\frac{2p_\mu p_\nu}{(p^2-a^2)^2} - \frac{2x(1-x)k_\mu k_\nu}{(p^2-a^2)^2} + \frac{2x(1-x)k^2 g_{\mu\nu}}{(p^2-a^2)^2} - \frac{g_{\mu\nu}}{p^2-a^2} - \frac{im\epsilon_{\mu\nu\alpha}k^\alpha}{(p^2-a^2)^2}\right\} \\ =&2ie^2\mu^\epsilon \int_0^1 dx \int \frac{d^np}{(2\pi)^{\frac{n}{2}}}\left\{\frac{2p_\mu p_\nu}{(p^2-a^2)^2} - \frac{g_{\mu\nu}}{(p^2-a^2)} \right. \\ &\left. + \left[\frac{2x(1-x)(k^2 g_{\mu\nu}-k_\mu k_\nu)}{(p^2-a^2)^2}\right] - \frac{im\epsilon_{\mu\nu\alpha}k^\alpha}{(p^2-a^2)^2}\right\}.\end{aligned} \tag{5}$$

To carry out separating $\Pi_{\mu\nu}(k)$ into three terms $\Pi_{\mu\nu}(k) = \Pi_{1\mu\nu}(k) + \Pi_{2\mu\nu}(k) + \Pi_{3\mu\nu}(k)$ and using the following formulae of the dimensional regularization :

$$I_o = \int \frac{d^np}{(2\pi)^{\frac{n}{2}}} \frac{1}{(p^2 - a^2)^\alpha} = \frac{i(-\pi)^{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \frac{\Gamma\left(\alpha - \frac{n}{2}\right)}{\Gamma(\alpha)} \frac{1}{(-\alpha^2)^{\left(\alpha - \frac{n}{2}\right)}}, \tag{6}$$

$$I_{\mu\nu} = \int \frac{d^np}{(2\pi)^{\frac{n}{2}}} \frac{p_\mu p_\nu}{(p^2 - a^2)^\alpha} = \frac{g_{\mu\nu}(-\alpha^2)}{\left(\alpha - 1 - \frac{n}{2}\right)} I_o, \tag{7}$$

$$\Gamma\left(2 - \frac{n}{2}\right) = \left(1 - \frac{n}{2}\right)\Gamma\left(1 - \frac{n}{2}\right), \quad \Gamma(2) = \Gamma(1) = 1, \tag{8}$$

we obtain:

$$\Pi_{1\mu\nu}(k) = 2ie^2\mu^\epsilon \int_0^1 dx \int \frac{d^np}{(2\pi)^{\frac{n}{2}}} \left[\frac{2p_\mu p_\nu}{(p^2 - a^2)^2} - \frac{g_{\mu\nu}}{(p^2 - a^2)}\right] = 0, \tag{9}$$

$$\begin{aligned}\Pi_{2\mu\nu}(k) &= 2ie^2\mu^\epsilon \int_0^1 2x(1-x)\left(k^2 g_{\mu\nu} - k_\mu k_\nu\right) dx \int \frac{d^np}{(2\pi)^{\frac{n}{2}}} \frac{1}{(p^2 - a^2)^2} \\ &= \frac{e^2\mu^\epsilon\left(k_\mu k_\nu - k^2 g_{\mu\nu}\right)}{(2\pi)^{-1/2}} \int_0^1 dx \frac{x(1-x)}{[m^2 - x(1-x)k^2]^{1/2}},\end{aligned} \tag{10}$$

$$\begin{aligned}\Pi_{3\mu\nu}(k) &= 2e^2 m\epsilon_{\mu\nu\alpha}\mu^\epsilon k^\alpha \int_0^1 dx \int \frac{d^np}{(2\pi)^{\frac{n}{2}}} \times \frac{1}{(p^2 - a^2)^2} \\ &= e^2 m\epsilon_{\mu\nu\alpha}\mu^\epsilon k^\alpha \frac{i\sqrt{\pi}}{\sqrt{2}} \int_0^1 dx \frac{1}{[m^2 - x(1-x)k^2]^{1/2}}.\end{aligned} \tag{11}$$

From the expression (10), we are easy to see that $\Pi_{2\mu\nu}(0) = 0$. So the final result, we find :

$$\Pi_{\mu\nu}(k)_{k^2=0} = [\Pi_{1\mu\nu}(k) + \Pi_{2\mu\nu}(k) + \Pi_{3\mu\nu}(k)]\,|_{k^2=0} \Rightarrow \Pi_{\mu\nu}(k)_{k^2=0} = \Pi_{3\mu\nu}(k)_{k^2=0} \neq 0. \tag{12}$$

The expression (12) talk to us that in the dimensional regularization method the photon have additional mass that is differential from zero, even its momentum equal zero.

In the next section, we will study this problem by Pauli-Villars regularization method.

3. Pauli-Villars regularization

Pauli-Villars regularization consists in replacing the singular Green's functions of the massive free field with the linear combination [4] :

$$\Delta(x) \to reg_M\Delta(m) = \Delta(m) + \sum_i c_i\Delta(M_i). \tag{13}$$

Here the symbol $\Delta_c(m)$ stands for the Green's function of the field of mass m, and the symbol $\Delta(M_i)$ are auxiliary quantities representing Green's function of fictitious fields with mass M_i, while c_i are certain coefficients satisfying special conditions. These conditions are chosen so that the regularized function $reg\Delta(x; m)$ considered in the configuration representation turns out to be sufficiently regular in the vicinity of the light cone, or (what is equivalent) such that the function $\bar{\Delta}(p; m)$ in the momentum representation falls off sufficiently fast in the region of large $|p|^2$.

On the base of Pauli-Villars regularization we calculate the polarization tensor operator in QED_3. For the vacuum polarization tensor we find the following expression :

$$\Pi^M_{\mu\nu}(k) = \frac{ie^2}{(2\pi)^{3/2}} \sum_i^{n_f} c_i \int d^3p \frac{Tr\left[\gamma_\mu\left(M_i + \hat{p} - \frac{1}{2}\hat{k}\right)\gamma_\nu\left(M_i + \hat{p} - \frac{1}{2}\hat{k}\right)\right]}{\left[M_i^2 + \left(p - \frac{1}{2}k\right)^2\right] \times \left[M_i^2 + \left(p + \frac{1}{2}k\right)^2\right]}, \tag{14}$$

with $c_o = 1; M_o = m; M_i = m\lambda_i; \sum_{i=0}^{n_f} c_i = 0; \sum_{i=0}^{n_f} c_i M_i = 0 \ (i = 1, 2, ...n_f)$.

For simplicity, but without loss of generality, we may choose both the electron mass and two mass of auxiliary fields to be positive, the coefficients λ_i ultimately go to infinity to recover the original theory.

$$\Pi^M_{\mu\nu}(k) = \frac{ie^2}{(2\pi)^{3/2}} \sum_i^{n_f} c_i \int d^3p \int_0^1 dx \frac{P(M_i)}{\left[M_i^2 - p^2 + (x^2 - x)k^2\right]^2}. \tag{15}$$

$$\begin{aligned} P(M_i) =& 2\Big\{ M_i^2 g_{\mu\nu} + 2p_\mu p_\nu + (1-2x)p_\nu k_\mu + 2(x^2 - x)k_\mu k_\nu \\ & - g_{\mu\nu}[p^2 + (1-2x)pk + (x^2 - x)k^2] - iM_i \epsilon_{\mu\nu\alpha} k^\alpha \Big\}. \end{aligned} \tag{16}$$

Neglecting the integrals that contain the odd terms of p in $P(M_i)$ we get:

$$\begin{aligned} \Pi^M_{\mu\nu}(k) = & \frac{ie^2}{(2\pi)^{3/2}} \sum_i^{n_f} c_i \int d^3p \int_0^1 dx \times \\ & \frac{2\left\{M_i^2 g_{\mu\nu} + 2p_\mu p_\nu + 2(x^2-x)k_\mu k_\nu - g_{\mu\nu}p^2 - 2g_{\mu\nu}\left(x^2 - x\right)k^2 - iM_i\epsilon_{\mu\nu\alpha}k^\alpha\right\}}{\left[M_i^2 - p^2 + (x^2 - x)k^2\right]^2}. \end{aligned} \tag{17}$$

The expression $\Pi^M_{\mu\nu}(k)$ can be written in the form:

$$\Pi^M_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Pi^M_1(k^2) + im\epsilon_{\mu\nu\alpha}k^\alpha \Pi^M_2(k^2) + g_{\mu\nu}\Pi^M_3(k^2). \tag{18}$$

Set $a_i^2 = M_i^2 + (x^2 - x)k^2 = M_i^2 - x(1-x)k^2$, we have:

$$\Pi^M_1(k^2) = 4ie^2 \sum_{i=0}^{n_f} c_i \int_0^1 x(1-x)dx \int \frac{d^3p}{(2\pi)^{3/2}} \times \frac{1}{\left(a_i^2 - p^2\right)^2}, \tag{19}$$

$$\Pi^M_2(k^2) = -\frac{2ie^2}{m} \sum_{i=0}^{n_f} c_i M_i \int_0^1 dx \int \frac{d^3p}{(2\pi)^{3/2}} \times \frac{1}{\left(a_i^2 - p^2\right)^2}, \tag{20}$$

$$\Pi^M_3(k^2) = 2ie^2 \sum_{i=0}^{n_f} c_i \left[\int_0^1 dx \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\left(a_i^2 - p^2\right)^2} + 2\int_0^1 dx \int \frac{d^3p}{(2\pi)^{3/2}} \frac{P^2}{\left(a_i^2 - p^2\right)^2}\right]. \tag{21}$$

If we carry out these integrations in the momentum space, it is straightforward to arrive: $\Pi^M_3(k^2) = 0$ as expected by the gauge invariance.

Here comes the crucial point: we can't blindly take only one auxiliary field with $M = \lambda m$ as usual; this choice is missionary conditions $\sum_{i=0}^{n_f} c_i = 0; \sum_{i=0}^{n_f} c_i M_i = 0$ must be matched. This is possible only fixing $\lambda = 1$. Thus, the number of regulators must be at leat two, otherwise we can't get the coefficients λ_i becoming arbitrarily large. So, let us take : $c_1 = \alpha - 1; c_2 = -\alpha; c_j = 0$ when $j > 2$.

Where the parameter α can assume any real value except zero and the unity, so that. condition (15) is satisfied. For $\lambda_1, \lambda_2 \to \infty$ and apply it to (18). To pay attention $\Gamma(1/2) = \sqrt{\pi}$, we have

$$\begin{aligned} \Pi^M_1 =& 4ie^2 \sum_{i=0}^{n_f} c_i \int_0^1 dx x(1-x) \int \frac{d^3p}{(2\pi)^{3/2}} \times \frac{1}{\left(a_i^2 - p^2\right)^2} \\ =& -\frac{e^2k^2}{(2\pi)^{-1/2}} \int_0^1 dx x(1-x)\left[\frac{c_o}{(a_o^2)^{1/2}} + \frac{c_1}{(a_1^2)^{1/2}} + \frac{c_2}{(a_2^2)^{1/2}}\right], \end{aligned} \tag{22}$$

where

$$a_o^2 = m^2 - x(1-x)k^2; a_1^2 = \lambda_1 m_1^2 - x(1-x)k^2; a_2^2 = \lambda_2 m_2^2 - x(1-x)k^2. \tag{23}$$

Thus , when $\lambda_1, \lambda_2 \to \infty$:

$$\Pi_1^M(k^2) \to \Pi_1(k^2) = -\frac{e^2k^2}{(2\pi)^{-1/2}} \int_0^1 dx x(1-x) \frac{1}{[m^2 - x(1-x)k^2]^{1/2}}, \tag{24}$$

and consequently, $\Pi_1(0) = 0$.

From (20), we have :

$$\begin{aligned} \Pi_2^M(k^2) =& \frac{e^2}{4m\pi} \int_0^1 dx \bigg\{ \frac{m}{[m^2 - x(1-x)k^2]^{1/2}} \\ &+ \frac{(\alpha-1)M_1}{[M_1^2 - x(1-x)k^2]^{1/2}} - \frac{(\alpha-1)M_2}{[M_2^2 - x(1-x)k^2]^{1/2}} \bigg\}. \end{aligned} \tag{25}$$

Taking the limit $\lambda_1, \lambda_2 \to \pm\infty$ (depending on couplings c_1 and c_2 having the same sign or different sign $\lambda \to +\infty$ or $\lambda \to -\infty$), for photon momentum k=0, yields:

- if $\lambda_1 \to +\infty; \lambda_2 \to \infty$: the couplings c_1, c_2 have the different sign, and $\alpha < 0$ or $\alpha > 0$, to

$$\Pi_2(0) = 0. \tag{26}$$

- if $\lambda_1 \to +\infty; \lambda_2 \to -\infty$: the couplings c_1, c_2 have the same sign, and $0 < \alpha < 1$

$$\Pi_2(0) = \frac{e^2}{\sqrt{2}m(\pi)^{3/2}}(1 + \alpha - 1 + \alpha) = \frac{2e^2\alpha}{\sqrt{2}m(\pi)^{3/2}}. \tag{27}$$

From the results (26) and (27), we can be written them in the form

$$\Pi_2(0) = \frac{\alpha e^2}{\sqrt{2}m\pi^{3/2}}(1 - s), \tag{28}$$

with $s = sign\left(1 - \frac{1}{\alpha}\right)$.

It is obvious that, from (28), we saw: if $0 < \alpha < 1$ and $s = -1$ the couplings c_1, c_2 have the same sign $\Pi_2(0) \neq 0$; in this case photon requires a topological mass, proportional to $\Pi_2(0)$, coming from proper insertions of the antisymmetry sector of the vacuum polarization tensor in the free photon propagator. If we assume that α is outside this range $(0, 1)$ and c_1 and c_2 have opposite signs and $\Pi_2(0) = 0$.

We then conclude that this arbitrariness α reflects in different values for the photon mass. The new parameter s may be identified with the winding number of homologically nontrivial gauge transformations and also appears in lattice regularization [7].

Now we face another problem: which value of α leads to the correct photon mass? A glance at equation (21) and we realize that $\Pi_2(k^2)$ is ultraviolet finite by naive power counting. We were taught that a closed fermion loop must be regularized as a whole so to preserve gauge invariance. However having done that we have affected a finite antisymmetric piece of the vacuum polarization tensor and, consequently, the photon mass. The same reasoning applies when, using Pauli-Villars regularization, we calculate the anomalous magnetic moment of the electron; again, if care is not taken, we may arrive at a wrong physical result.

In order to get of this trouble we should pick out the value of α that cancels the contribution coming from the regulator fields. From expression (28), we easily find that this occurs for $(c_1 = c_2)$ because in this case the signs of the auxiliary masses are opposite, in account of condition (15). From (28), we obtain $\Pi_2(0) = \frac{e^2}{\sqrt{2}m\pi^{3/2}}$, in agreement with the other approach already mentioned. We should remember that Pauli Villars regularization violated party symmetry $(2+1)$ dimensions. Nevertheless, for this particular choice α, this symmetry is restored as regulator mass get larger and larger.

	$\alpha < 0, \alpha > 1$	$0 < \alpha < 1$	photon mass
c_1 and c_2 opposite sign	$\Pi_2(0) = 0$		equal zero
c_1 and c_2 same sign		$\Pi_2(0) \neq 0$	unequal zero
$c_1 = c_2; \alpha = \frac{1}{2}$			$\Pi_2(0) = -\frac{e^2}{\sqrt{2}m\pi^{3/2}}$

4. Conclusion

In depending on sign of the couplings c_1 and c_2, same and opposite sign the Pauli-Villars regularization give a result $\Pi_2(0) = \frac{\alpha e^2}{\sqrt{2}m\pi^{3/2}}(1-s)$, where $s = sign\left(1 - \frac{1}{\alpha}\right)$. Results obtained by regularization Pauli-Villars and dimensional methods are coincided if the gauge invariance of theory is considered carefully step by step in these calculations. When $c_1 = c_2; \alpha = \frac{1}{2}$, the expressions obtained by the Pauli-Villars and the dimensional method have same results $\Pi_2(0) = \frac{e^2}{\sqrt{2}m\pi^{3/2}}$ in QED_3 in agreement with the other approaches for these problems [6].

Acknowledgments. This work was supported by Vietnam National Research Programme in National Sciences N406406.

References

[1] S. Dese, R. Jackiw, S. Templeton, *Ann. of Phys.* 140 (1982) 372.
[2] R. Delbourggo, A.B. Waites, *Phys. Lett.* B300 (1993) 241.
[3] B.M. Pimentel, A.T. Suzuki, J.L. Tomazelli, *Int. J. Mod. Phys.* A7 (1992) 5307.
[4] N.N. Bogoliubov, D.V. Shirkov, *Introduction to the Theory of Quantumzed Fields*, John Wiley-Sons, New York, 1984.
[5] J.M. Jauch, F. Rohrlich, *The Theory of Photons and Electrons*, Addison-Weslay Publishing Company, London, 1955.
[6] B.M. Pimentel, A.T. Suzuki, J.L. Tomazelli, *Int. J. Mod. Phys.* A7 (1992) 5307.
[7] A. Coste, M. Luscher, *Nucl. Phys.* 323 (1989) 631.